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Doctoral Thesis

The Phase Space for the Einstein-Yang-Mills Equations, Black Hole Mechanics, and a Condition for Stationarity

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Declaration of Authorship

I, Stephen M$^{\text{c}}$Cormick, declare that this thesis titled, "*The Phase Space for the Einstein-Yang-Mills Equations, Black Hole Mechanics, and a Condition for Stationarity*" and the work presented in it are my own. I confirm that:

- This work was done wholly while in candidature for a research degree at Monash University.
- No part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.

A poet once said, "The whole universe is in a glass of wine." We will probably never know in what sense he meant that, for poets do not write to be understood. But it is true that if we look at a glass of wine closely enough we see the entire universe. There are the things of physics: the twisting liquid which evaporates depending on the wind and weather, the reflections in the glass, and our imagination adds the atoms. The glass is a distillation of the Earth's rocks, and in its composition we see the secrets of the universe's age, and the evolution of stars. What strange arrays of chemicals are in the wine? How did they come to be? There are the ferments, the enzymes, the substrates, and the products. There in wine is found the great generalization: all life is fermentation. Nobody can discover the chemistry of wine without discovering, as did Louis Pasteur, the cause of much disease. How vivid is the claret, pressing its existence into the consciousness that watches it! If our small minds, for some convenience, divide this glass of wine, this universe, into parts — physics, biology, geology, astronomy, psychology, and so on — remember that Nature does not know it! So let us put it all back together, not forgetting ultimately what it is for. Let it give us one more final pleasure: drink it and forget it all!

Richard Feynman

MONASH UNIVERSITY

Abstract

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The Phase Space for the Einstein-Yang-Mills Equations, Black Hole Mechanics, and a Condition for Stationarity

Stephen M^c^Cormick

In this thesis, we study the phase space for the Einstein-Yang-Mills equations on an asymptotically flat manifold. The phase space is defined as a Hilbert manifold, which is modeled on weighted Sobolev spaces. We use an implicit function theorem argument to prove that the space of solutions to the constraint equations is a Hilbert submanifold of the phase space; this is equivalent to the statement that the Einstein-Yang-Mills constraints on an asymptotically flat manifold are linearisation stable.

It is then shown that the energy, momentum, charge and angular momentum are smooth maps acting on the constraint submanifold. This framework allows us to prove that the first law of black hole mechanics provides a condition for initial data to be stationary, in two distinct cases: when the Cauchy surface has an interior boundary, and when it does not. Both cases are established using a Lagrange multipliers argument.

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Contents

Conventions

We use the signature convention, $(-,+,+,+)$, whenever we discuss Lorentzian metrics.

We use the following conventions for indices on tensors considered over different bundles:

3-dimensional manifold	Latin lower case, mid-alphabet	$i, j, ...$
4-dimensional manifold	Greek lower case, mid-alphabet	$\mu, \nu...$
n-dimensional Lie algebra	Latin lower case, early alphabet	$a, b...$
$(4+n)$-dimensional principal bundle	Greek lower case, early alphabet	$\alpha, \beta...$

By an abuse of notation, we will write $\xi^\alpha = (\xi^0, \xi^i, \xi^a) = (\xi^\mu, \xi^a)$ to indicate an object that is to be identified with a tensor over the $(4+n)$-dimensional bundle, and use the index to distinguish between components. For example, we will consider ξ^0 as a scalar function ξ^i as a vector field over the related 3-manifold.

The second fundamental form of a spacelike hypersurface is $K_{ij} = \nabla_i n_j$, where n^j is the future pointing unit normal. This sign convention differs to that used in numerical relativity, but appears to be more prevalent in the geometric analysis community.

Chapter 1

Introduction

There is no certainty in sciences where one of the mathematical sciences cannot be applied, or which are not in relation with these mathematics.

Leonardo da Vinci

One of the most remarkable discoveries in the history of general relativity is the duality between the laws of thermodynamics and the laws of black hole mechanics [8, 14, 15, 39]. It is nearly impossible to discuss many interesting ideas in modern physics without at least a mention of the laws of black hole mechanics: Hawking radiation, the information paradox and the recent ideas surrounding black hole firewalls, to name a few. Through this duality, we are able to discuss the thermodynamic properties, such as entropy and temperature, of a black hole.

In 1992, Sudarsky and Wald [57] discussed the first law of black hole mechanics in the context of Einstein-Yang-Mills theory. Among other things, they noted that certain surface integrals, associated with the Hamiltonian, were closely related to the first law. It was argued that the differential relationship pertaining to the first law should provide a condition for stationarity of the Einstein-Yang-Mills equations. Their argument is based on earlier work by Brill, Deser and Fadeev [17], who proposed a condition for stationarity in the pure Einstein case.

Brill, Deser and Fadeev argued that stationary solutions were exactly those solutions that extremise the ADM mass on the space of solutions. Both arguments were based on the method of Lagrange multipliers, however neither provided the mathematical machinery required to make such an argument rigorous. The essential missing ingredient, to develop this argument into a mathematical proof, is a manifold structure for the space of solutions.

In 2005, Bartnik provided such a Hilbert manifold structure for the space of solutions to the Einstein constraints, and from this a complete proof of the Brill, Deser and Fadeev argument was given [11]. At first this appears to contradict the argument of Sudarsky and Wald, since we have that a solution is stationary if and only if the condition $dm = 0$ is satisfied. However, the case considered by Bartnik has no Maxwell or Yang-Mills fields, and the initial data manifold has a single asymptotic end with no interior boundary, so the first law reduces to $dm = 0$.

This thesis provides a Hilbert manifold structure for the space of solutions to the Einstein-Yang-Mills constraints, and establishes an analogous condition for stationarity in this case. We then examine the case where the initial data manifold has a closed 2-surface interior boundary, and demonstrate that the usual expression of the first law gives a condition for stationarity. The Hilbert manifold structure provided for the space of solutions is equivalent to the condition of linearisation stability, which was studied in detail in the 1970s for the case where the manifold was compact without boundary [1, 2, 32]. That is, the results here establish linearisation stability for the Einstein-Yang-Mills constraints on an asymptotically flat manifold.

The outline of this thesis is as follows. Chapter 2 gives background and context to this work; we introduce the initial data formulation of general relativity and the Einstein-Yang-Mills equations, then briefly discuss the laws of black hole mechanics and the property of linearisation stability. Chapter 3 provides a short summary of known results pertaining to the weighted Sobolev spaces that the phase space Hilbert manifold is modeled on. The weighted inequalities contained therein are the basis of many of the estimates to follow. The crux of the thesis begins with Chapter 4, where we construct the phase space and prove that the space of solutions to the constraints is a Hilbert submanifold of this phase space. The relevant definitions of mass, charge and angular momentum are collected in Chapter 5 and we prove they are smooth functions on the constraint submanifold. In Chapter 6, we finally get to the first law of black hole mechanics where two cases are discussed separately: when horizon terms are present, and when they are not. In both cases, a new Hamiltonian is introduced à la Regge and Teitelboim [50], which behaves as a Lagrange function. Then a Lagrange multipliers argument demonstrates that the first law gives a condition for stationarity.

Chapter 2

Background

But what is your final goal, you may ask. That goal will become clearer, will emerge slowly but surely, much as the rough draft turns into a sketch, and the sketch into a painting through the serious work done on it, through the elaboration of the original vague idea and through the consolidation of the first fleeting and passing thought.

Vincent van Gogh

2.1 The Einstein Constraint Equations

The Einstein field equations (2.1) are a system of ten partial differential equations (PDEs), relating matter to the curvature of spacetime, and are the core of general relativity. The source matter is described by a symmetric two-tensor, $T_{\mu\nu}$, known as the stress-energy tensor, and one solves the field equations,

$$^4R_{\mu\nu} - \frac{1}{2}{}^4R\,{}^4g_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{2.1}$$

for a Lorentzian metric, $^4g_{\mu\nu}$, on a differentiable 4-manifold, ^4V. In the above, $^4R_{\mu\nu}$ is the Ricci curvature and ^4R is the scalar curvature of ^4g.

Similar to other physical theories, the Einstein equations can be recast as an initial data problem in the form of a Hamiltonian system. This is particularly useful when one is concerned with the space of solutions, or finding particular solutions, as it reduces the number of equations from ten down to four constraints. This also allows one to work with Riemannian geometry, rather than Lorentzian geometry. The idea of a Hamiltonian formulation of general relativity gained significant popularity when Dirac [29, 30] lay groundwork in the pursuit of a quantum theory of gravity. While half a century later we are still without a quantum theory of gravity, the Hamiltonian formulation has found many uses in the study of classical general relativity. The Hamiltonian framework generally used now is due to Arnowitt, Deser and Misner [3, 4, 6], and is closely related to the mass definition given by the same authors [5]. This formalism and mass definition is known as the ADM formalism and ADM mass respectively. The ideas behind the ADM formalism are outlined below (see also [6, 38]).

In order to recast the field equations (2.1) as an initial data problem, we must first interpret "time evolution" in the context on general relativity; that is, we must split spacetime into "space" and "time". Let $({}^4V, {}^4g)$ be a globally hyperbolic spacetime, so that it can be foliated by spacelike hypersurfaces, ${}^4V = \bigcup_\tau M_\tau$, where each $M_\tau = t^{-1}(\tau)$ is the τ level surface of some time function, t. The time evolution vector is then a future-pointing timelike vector t^μ, such that $t^\mu dt_\mu = 1$, and time derivatives are interpreted as Lie derivatives with respect to t^μ.

Given a spacelike hypersurface, M_{τ_0}, initial data for the Einstein equations is the induced 3-metric, g, and second fundamental form, K, on M_{τ_0}. We also make use of the decomposition $t^\mu = Nn^\mu + X^\mu$, where n is the future-pointing unit normal to M_{τ_0} and X is tangential to M_{τ_0}. This allows us to view t^μ as data on M_{τ_0}. The quantities N and X are called the lapse function and shift vector respectively and we often call t^μ the lapse-shift 4-vector.

If ${}^4g$ is to satisfy the field equations (2.1) then obviously g and K cannot be arbitrary, and in fact must also satisfy some geometric constraints. The Gauss and Codazzi equations can be used to write the curvature of g in terms of the curvature of ${}^4g$ and K. From this it can be shown that a solution to the field equations must impose the following constraints.

$$R(g) - K^{ij}K_{ij} + (K^i_i)^2 = 16\pi T_{00}, \tag{2.2}$$

$$\nabla^j K_{ij} - \nabla_i(K^j_j) = 8\pi T_{0i}, \tag{2.3}$$

where $R(g)$ is the scalar curvature of the induced metric and the zero index refers to a projection of the stress-energy tensor normal to the initial data surface. The remaining Einstein equations give the evolution equations,

$$\frac{\partial}{\partial t} g_{ij} = \mathcal{L}_X g_{ij} - 2NK_{ij}, \tag{2.4}$$

$$\begin{aligned}\frac{\partial}{\partial t} K_{ij} =& \mathcal{L}_X K_{ij} - \nabla_i\nabla_j N + N\big(R_{ij} + K^k_k K_{ij} \\ & - 2K_{ik}K^k_j + 4\pi[T^\mu_\mu g_{ij} - 2T_{ij}]\big),\end{aligned} \tag{2.5}$$

where T is again the source stress-energy tensor and we use $\mathcal{L}$ to denote the Lie derivative. Note that the lapse and shift are not constrained by these equations; they simply correspond to a coordinate choice, and are freely specifiable.

Now consider the constraint equations (2.2), (2.3) on an arbitrary 3-manifold, $\mathcal{M}$, where g is a Riemannian metric and K is simply some symmetric covariant 2-tensor. One naturally may ask if it is then possible to find a unique solution to the Einstein equations, in which $(\mathcal{M}, g)$ embeds isometrically with K as the second fundamental form. Remarkably, for suitably regular (g, K), this has been answered in the affirmative. In 1952, Choquet-Bruhat (then Foures-Bruhat) [35] established the existence result, and uniqueness was settled in her joint work with Geroch [23] in 1969. For this reason, one may consider solutions to the constraints rather than solving the full Einstein equations when looking for solutions. The weakest regularity assumptions on the initial data to ensure the Cauchy problem is well-posed, is due to recent work of Klainerman, Rodnianski and Szeftel [44]. This requirement for well-posedness is $R(g), \nabla(K) \in L^2_{loc}$, which may be guaranteed by imposing the condition $(g, K) \in W^{2,2}_{loc} \times W^{1,2}_{loc}$.

If one were to interpret the 3-metric as the position variable in the Hamiltonian formulation, then the associated momentum variable is given by

$$\pi^{ij} := (K^{ij} - K^k_k g^{ij})\sqrt{g}. \tag{2.6}$$

The phase space for the Einstein equations is given by pairs (g, π), with suitable regularity and asymptotics prescribed. Hamiltonians will be discussed in greater detail in Chapter 6.

2.2 The Einstein-Yang-Mills Equations

The matter, serving as the source for the field equations, is also governed by a system of equations. Since these equations are defined on a curved spacetime, the matter fields themselves are affected by the geometry of the spacetime; in this case, we say the systems of equations are coupled. For example, both Maxwell's equations, and the stress energy tensor for electromagnetism in curved spacetime, depend on the metric. This corresponds to photons being affected by the curvature, while the energy of the electromagnetic field causes curvature itself. Here we are interested in the case where the matter source comes from Yang-Mills fields, a generalisation of the electromagnetic field.

Electromagnetism is a gauge theory with a $U(1)$ symmetry, where Yang-Mills theory is a gauge theory with any compact Lie group as the gauge group. Introduced in 1954 by Chen Ning Yang and Robert Mills [60], Yang-Mills theory is of particular interest to physicists as the standard model of particle physics is a quantum Yang-Mills theory, with $U(1) \times SU(2) \times SU(3)$ gauge symmetry. There has been a significant interest in coupling Yang-Mills fields to gravity since the discovery of the Bartnik-McKinnon particle-like solutions [12, 55] and coloured black holes [16, 54], as there are no analogous solutions for the Einstein-Maxwell system.

Let G be some compact Lie group with Lie algebra, $\mathfrak{g}$; this will be the gauge group for the Yang-Mills fields. The geometric structure describing Yang-Mills fields is a principal G-bundle, P, over the spacetime, ${}^4V$, equipped with a connection one-form, ω. We can pullback ω, via a local section $\iota : U \to P$, to obtain a $\mathfrak{g}$-valued one-form on $U \subset {}^4\mathcal{M}$,

$$ {}^4A := \iota^*(\omega). \tag{2.7} $$

This quantity is called the *gauge potential* and describes the Yang-Mills field in the gauge specified by the choice of ι. Another important quantity in Yang-Mills theory is the curvature of the connection, given by

$$ \Omega := d\omega + \omega \wedge \omega, \tag{2.8} $$

which is pulled back via ι to the *field strength tensor*,

$$ F := \iota^*(\Omega), \tag{2.9} $$

a $\mathfrak{g}$-valued two-form on U. In local coordinates on ${}^4\mathcal{M}$, we write

$$ F^a_{\mu\nu} = \nabla_\mu A^a_\nu - \nabla_\nu A^a_\mu + C^a_{bc} A^b_\mu A^c_\nu, \tag{2.10} $$

where ∇ is the Levi-Civita connection of g, and C^a_{bc} are the structure constants of $\mathfrak{g}$, which are defined by the Lie bracket: $[\theta, \chi]^a = C^a_{bc}\theta^b\chi^c$ for all $\theta, \chi \in \mathfrak{g}$. Note that ∇ can be replaced with any torsion-free connection in (2.10), as $F_{\mu\nu}$

is an antisymmetric tensor. Since G is compact, $\mathfrak{g}$ is the direct sum of semi-simple and Abelian Lie algebras. It follows that the negative of the Killing form provides an adjoint-invariant positive definite inner product on the semi-simple factor, while the usual Euclidean inner product suffices on the Abelian factor. This inner product, which we denote by γ, is used to identify $\mathfrak{g}$ with $\mathfrak{g}^*$, the Lie coalgebra.

In coordinates, the source-free Yang-Mills equations are given by

$$g^{\rho\mu}(\nabla_\rho F^a_{\mu\nu} + C^a_{bc} A^b_\rho F^c_{\mu\nu}) = 0. \tag{2.11}$$

The Yang-Mills equations are often considered on a fixed background, in which case g in (2.11) is fixed. However, by inserting the Yang-Mills stress-energy tensor,

$$T^{\mu\nu} = \frac{1}{4\pi}(\gamma^{ab} F_a^{\mu\rho} F_b{}^\nu{}_\rho - \frac{1}{4} g^{\mu\nu} F_a^{\rho\sigma} F^a_{\rho\sigma}), \tag{2.12}$$

into the Einstein equations as the source term, and insisting that g in (2.11) solves the field equations, the equations become coupled. This coupled system, (2.1) and (2.11) with T given by (2.12), form the Einstein-Yang-Mills equations.

Initial data for the Yang-Mills equations is a pair (A, ε), where the Hamiltonian position variable, A, is the orthogonal projection of ${}^4A$ onto the initial data surface, and the associated canonical momentum variable is

$$\varepsilon^i_a = -4E^i_a\sqrt{g} = -4F^{0i}_a\sqrt{g};$$

negative four times the Yang-Mills electric field density, as viewed by a Gaussian normal set of observers for the initial data slice. We will use both E and ε throughout this thesis, as it will often be more illuminating to express things in terms of the electric and magnetic fields, rather than A and ε. The Yang-Mills magnetic field density, B, also viewed by a Gaussian normal set of observers, is defined analogously to the electromagnetic case. We write

$$B_a^i = \frac{1}{2}\gamma_{ab}\epsilon^{ijk}F_{jk}^b = \frac{1}{2}\gamma_{ab}\epsilon^{ijk}(\nabla_j A_k^b - \nabla_k A_j^b + [A_j, A_k]^b), \tag{2.13}$$

where ϵ is the Levi-Civita symbol; a completely antisymmetric tensor density of weight 1. The Yang-Mills equations impose additional constraints on the initial data, analogous to the Gauss law in electromagnetism:

$$\nabla_j E_a^j + C_{ab}^c A_j^b E_c^j = 4\pi\rho_a, \tag{2.14}$$

where the source term, ρ, is a $\mathfrak{g}$-valued function on $\mathcal{M}$, corresponding to the Yang-Mills electric charge density.

The Yang-Mills initial data evolves according to

$$\frac{\partial}{\partial t}A_i^a = \mathcal{L}_X A_i^a + \frac{1}{4}N\varepsilon_i^a g^{-1/2} - \nabla_i V^a - C_{bc}^a A_i^b V^c, \tag{2.15}$$

$$\frac{\partial}{\partial t}\varepsilon_a^i = \mathcal{L}_X \varepsilon_a^i - \epsilon^{ijk}\big(\nabla_j(NB_{ak}g^{-1/2}) + C_{abc}NB_k^c A_j^b g^{-1/2}\big) - C_{ca}^b \varepsilon_b^i V^c, \tag{2.16}$$

where we have assumed no 3-current source term. By inserting the Yang-Mills stress energy tensor (2.12) into (2.2) - (2.5), the constraint and evolution equations for the coupled system are obtained.

In the published literature, the weakest regularity assumptions to ensure that the Yang-Mills Cauchy problem on a curved background is well-posed, are due to Chruściel and Shatah [24]. Specifically, it is required that the initial data have local regularity, $(A, E) \in H^3_{loc} \times H^2_{loc}$. However, a recent preprint of Ghanem [36] improves this to the same regularity required for data on a Minkowski background, $(A, E) \in H^2_{loc} \times H^1_{loc}$ [31, 43].

The phase space for the Einstein-Yang-Mills equations considered here is tuples, (g, A, π, ε), with local regularity $H^2 \times H^2 \times H^1 \times H^1$ and appropriately prescribed asymptotics. This is, we have sufficient regularity for each of the Einstein equations and Yang-Mills equations to have well-posed Cauchy problems, so one would expect that the coupled system is also well posed; however, to the best of the author's knowledge this has yet to be explicitly demonstrated.

Throughout this work we primarily concern ourselves with the constraint equations and make very little reference to the underlying G-bundle.

2.3 Black Hole Mechanics

It is universally known that entropy is a non-decreasing function of time. As such, a system with high entropy collapsing to a black hole, which is characterised by only a handful of parameters, may give pause. However, this apparent paradox is resolved by the advent of black hole thermodynamics and the remarkable results of Bardeen, Carter, Hawking [8] and Bekenstein [14]. Hawking's area theorem states that the surface area of spacelike cross-sections of a black hole's event horizon is non-decreasing with time. Bekenstein then proposed a generalised second law [15] of thermodynamics, stating that the usual thermodynamic entropy plus a multiple of the horizon area is non-decreasing with time. This new measure of entropy now bears both of their names: Bekenstein-Hawking entropy.

By considering quantum field theory on a black hole background, Hawking demonstrated another remarkable property of black holes; namely, that they actually radiate energy [39]. A black hole emits electromagnetic radiation as would a black body, with temperature proportional to the surface gravity of the black hole. It may seem tenuous to call this a duality between the laws of thermodynamics and black hole mechanics, but the relationship is much stronger than this. Bardeen, Carter and Hawking described the following four laws of black hole mechanics entirely analogous to the regular laws of thermodynamics:

The Zeroth Law of Thermodynamics. *A system in equilibrium has constant temperature throughout.*

The Zeroth Law of Black Hole Mechanics. *A stationary black hole has constant surface gravity.*

The First Law of Thermodynamics. *For any perturbation to a system in equilibrium, the following differential relationship is satisfied:*

$$dE = TdS - PdV, \tag{2.17}$$

where E is the energy, T is the temperature, S is the entropy, P is the pressure and V is the volume.

The First Law of Black Hole Mechanics. *For any perturbation to a stationary black hole, the following differential relationship is satisfied:*

$$dm = \frac{\kappa}{8\pi} dA + \Omega dJ + V \cdot dQ, \tag{2.18}$$

where m is the mass, κ is the surface gravity, A is the horizon area, Ω is the angular velocity, J is the angular momentum, V is the electric potential and Q is the electric charge of the black hole.

The Second Law of Thermodynamics. *The entropy of an isolated system never decreases.*

The Second Law of Black Hole Mechanics. *The horizon area of a black hole never decreases.*

The Third Law of Thermodynamics. *It is impossible to reduce the temperature of a system to zero in finite time.*

The Third Law of Black Hole Mechanics. *It is impossible to reduce the surface gravity of a black hole to zero in finite time.*

All four laws are included to motivate the duality, however it is the first law that is the main focus of this work; the zeroth law is also briefly mentioned in Chapter 6. When the laws are stated, it is often implicit that one is considering only pure Einstein black holes or Einstein-Maxwell black holes. However, the laws as stated above, are unchanged in the Einstein-Yang-Mills case, which is the case considered in this work. In fact, there are also versions of these laws when gravity is coupled to other fields [40, 52].

2.4 Linearisation Stability

The property of linearisation stability is concerned with the validity of first order perturbation theory. Suppose $F(x) = c$ is a system of equations with a known solution, $x = x_0$. First-order perturbation theory is a method to give approximate solutions nearby x_0. If y is a solution to the linearised equation $DF_x[y] = 0$ then $x = x_0 + \epsilon y$ is expected to be an approximate solution to F(x)=c, for small epsilon. The system $F(x) = c$ is said to be linearisation stable at x_0 if for every solution, y, to the linearised equation, there exists a curve of solutions $x(t)$ with $x(0) = x_0$ and $x'(0) = y$. This is equivalent to the condition that the level sets, $F^{-1}(c)$, have a manifold structure.

The first-order perturbation theory is often used in place of the full Einstein equations to describe weak gravitational fields such as gravitational waves [34], or may be used to discuss stability of solutions under perturbations [51]. Naturally, it is then important to ensure that the Einstein equations are indeed linearisation stable.

Linearisation stability was first established for perturbations to Minkowski space by Choquet-Bruhat and Deser in 1973 [22]. Then in the same year, Fischer and Marsden established linearisation stability for generic solutions with a compact Cauchy surface [32], however linearisation stability fails for some exceptional solutions. These exceptional solutions were subsequently shown by Moncrief to be exactly when the Cauchy development contains a Killing vector field [48]. These results were then extended by Arms to the Einstein-Maxwell [1] and Einstein-Yang-Mills [2] cases, in 1977 and 1979 respectively.

Linearisation stability has also been studied in many other contexts, including gravity coupled to self-gravitating scalar fields [53] and cosmological models [18, 19]. Of particular interest, is the case where the Cauchy surface is an asymptotically flat manifold, as these solutions represent isolated gravitating systems. This case was not established until 2005 by Bartnik [11][1], and unlike compact case, the asymptotically flat case is linearisation stable at all solutions.

The results in Chapter 4 prove linearisation stability for the Einstein-Maxwell and Einstein-Yang-Mills constraints on an asymptotically flat manifold. This essentially fills a gap in the collection of works by Fischer, Marsden, Arms and Bartnik.

[1]At the time of publication, existence and uniqueness results for the regularity class used in Bartnik's work was not available, however it was mentioned that this work would imply linearisation stability given such results were established. The recent resolution of the bounded L^2 curvature conjecture [44] provides exactly such existence and uniqueness results.

Chapter 3

Weighted Sobolev Spaces

The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve.

Eugene Wigner

It is well known that Sobolev spaces are a natural setting for the study of PDEs. Unfortunately, several key properties of Sobolev spaces on bounded domains do not carry over to unbounded domains. Let $\mathring{W}^{k,p}(\Omega)$ be the completion of $C_c^\infty(\Omega)$ with respect to the usual $W^{2,2}$ norm, where $\Omega \subset \mathbb{R}^n$ is bounded with smooth boundary. It is well-known that the Laplacian has nice mapping properties bounded domains; however, the map $\Delta : \mathring{W}^{2,2}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ does not even have closed range.

The weighted Sobolev and Lebesgue spaces, introduced by Cantor [20], are often more appropriate spaces for the study of PDEs on unbounded domains; for example, when considered as a map between these spaces, the Laplacian is indeed Fredholm, $\Delta : W^{2,2}_\delta(\mathbb{R}^n) \to L^2_{\delta-2}(\mathbb{R}^n)$ [21, 47]. This fact will be particularly important in the proof of Theorem 4.15. In this Chapter, we introduce the weighted Lebesgue (L^p_δ) and Sobolev ($W^{k,p}_\delta$) spaces, and some useful results pertaining to them.

The definitions and results presented here are for weighted spaces on $\mathbb{R}^n$, however this is simply for the sake of presentation. As the weights are only significant near infinity, these results clearly hold for an asymptotically flat manifold in the sense of Chapter 4. Defining the spaces on $\mathbb{R}^n$ simply allows us to reserve the introduction of asymptotically flat manifolds for the following Chapter. Most of the results presented here are well-known and are frequently used in the study of asymptotically flat manifolds. It is useful to collect these results together in one place for reference.

Define the weighted norms,

$$\|u\|_{p,\delta} = \begin{cases} \left(\int_{\mathbb{R}^n} |u|^p \, r^{-\delta p - n} dx^n\right)^{1/p}, & p < \infty \\ \operatorname*{ess\,sup}_{\mathbb{R}^n}(r^{-\delta}|u|), & p = \infty \end{cases} \tag{3.1}$$

$$\|u\|_{k,p,\delta} = \sum_{|\alpha|=0}^{k} \|\partial^\alpha u\|_{p,\delta-|\alpha|}, \tag{3.2}$$

where $r(x) = \sqrt{1+|x|^2}$ and α is a multi-index, in the usual sense. The spaces $L^p_\delta(\mathbb{R}^n)$ and $W^{k,p}_\delta(\mathbb{R}^n)$ are then defined as the completion of $C^\infty_c(\mathbb{R}^n)$ with respect to these norms, respectively. Weighted spaces of sections of bundles are defined in the usual way. Intuitively, these spaces are function spaces with local regularity L^p and $W^{k,p}$, which behave like $o(r^\delta)$ at infinity and whose derivatives decay appropriately. We will often omit reference to $\mathbb{R}^n$, the manifold, or the bundle that we are taking sections of, and simply write the spaces as $W^{k,p}_\delta$ when there is no risk of confusion.

The following weighted versions of standard inequalities (see, for example, [9, 21]) will be used frequently throughout this thesis:

Proposition 3.1 (Lebesgue embedding). *If* $1 \leq p \leq q \leq \infty$, $\delta_2 < \delta_1$ *and* $u \in L^q_{\delta_2}$, *then*

$$\|u\|_{p,\delta_1} \leq c\,\|u\|_{q,\delta_2}\,, \tag{3.3}$$

and therefore $L^q_{\delta_2} \subset L^p_{\delta_1}$.

Proposition 3.2 (Hölder's inequality). *If* $u \in L^q_{\delta_1}$, $v \in L^s_{\delta_2}$ *and* $\delta = \delta_1 + \delta_2$, $1 \leq p, q, s \leq \infty$, *then*

$$\|uv\|_{p,\delta} \leq \|u\|_{q,\delta_1}\,\|v\|_{s,\delta_2}\,, \tag{3.4}$$

where $1/p = 1/q + 1/s$.

Proposition 3.3 (Interpolation inequality). *For any* $\epsilon > 0$, *there is a constant,* $c(\epsilon)$, *such that for all* $u \in W^{2,p}_\delta$,

$$\|u\|_{1,p,\delta} \leq \epsilon\,\|u\|_{2,p,\delta} + c(\epsilon)\,\|u\|_{p,\delta}\,, \tag{3.5}$$

for $1 \leq p \leq \infty$.

Proposition 3.4 (Sobolev inequality). *If* $u \in W^{k,p}_\delta$, *then*

$$\|u\|_{np/(n-kp),\delta} \leq c\,\|u\|_{k,q,\delta} \tag{3.6}$$

for q *satisfying* $p \leq q \leq np/(n-kp)$.

If $kp > n$ *then*

$$\|u\|_{\infty,\delta} \leq c\|u\|_{k,p,\delta}. \tag{3.7}$$

Remark 3.5. The four inequalities above are all valid when the integrals, defining the norms, are taken over bounded domains.

Proposition 3.6 (Poincaré inequality)**.** *For any* $u \in W^{1,p}_{\delta}$, $\delta < 0$, *we have*

$$\|u\|_{p,\delta} \leq c\|\partial u\|_{p,\delta-1}. \tag{3.8}$$

There is also a weighted version of the Rellich compactness theorem, which we require for the proof of Theorem 4.15.

Proposition 3.7 (Rellich compactness)**.** *For* $k_1 > k_2$, $\delta_1 < \delta_2$ *and* $1 \leq p < \infty$, *the inclusion* $W^{k_1,p}_{\delta_1} \subset W^{k_2,p}_{\delta_2}$ *is compact.*

We also need some results that are specifically tailored to our particular problem. When integrating a divergence over the whole manifold, the divergence theorem gives boundary integrals "at infinity", which are to be understood in the sense of a limit. While some of these boundary integrals will correspond to physically relevant quantities, others are essentially irrelevant. The following Proposition from [11] allows us to control these terms that are not physically relevant.

Let B_R be the open ball of radius R centred at zero and define $E_R = \mathbb{R}^n \setminus \overline{B_R}$, $A_R = B_{2R} \setminus \overline{B_R}$ and $S_R = \overline{B_R} \setminus B_R$.

Proposition 3.8. *Suppose* $u \in W^{1,2}_{-3/2}(E_{R_0})$ *for some* $R_0 \geq 1$. *Then* $u \in L^1(S_R)$ *for every* $R \geq R_0$, *and there is a constant* c, *independent of* R, *such that*

$$\oint_{S_R} |u| \leq c\sqrt{R}\|u\|_{1,2,-3/2:A_R}, \tag{3.9}$$

where the notation in (3.9) indicates that the norm is taken over A_R.

‘

Chapter 4

The Phase Space for the Einstein-Yang-Mills Equations and the Constraint Submanifold

The world is full of obvious things which nobody by any chance ever observes.

Sherlock Holmes

Now that we have discussed the relevant background and machinery, we are able to get to the crux of the thesis. We begin by introducing the phase space for the Einstein-Yang-Mills equations. The results here extend Bartnik's work on the phase space for the Einstein equations on an asymptotically flat manifold [11], to the Einstein-Yang-Mills case.

Alternatively, this Chapter may be viewed as an extension of Arms' work on the linearisation stability of the Einstein-Maxwell [1] and Einstein-Yang-Mills [2] equations; where Arms considered the constraint equations on a compact manifold, we consider the asymptotically flat case. Linearisation stability is briefly discussed at the end of the Chapter.

4.1 The Phase Space

We begin by defining the 3-manifold, on which our initial data is to be defined. Let $\mathcal{M}$ be a paracompact, connected, non-compact 3-manifold without boundary. Further suppose that there exists a compact set, K, such that $\mathcal{M} \setminus K = \bigcup_{n=1}^{N} M_n$, where each M_n is diffeomorphic to $\mathbb{R}^3$ minus the closed unit ball. That is, there exists a collection of diffeomorphisms, $\phi_n : M_n \to \overline{B_1(0)} \in \mathbb{R}^3$. We call each M_n an *asymptotic end* of $\mathcal{M}$.

Fix a smooth background metric, $\mathring{g}$, such that $\mathring{g} = \phi_n^*(g_{\mathbb{R}^3})$, the pullback of the Euclidean metric, on each M_n. Further, fix a smooth function, $r \geq 1$, with $r = |x|$ on each M_n. A Riemannian manifold, $(\mathcal{M}, g)$, with $\mathcal{M}$ as above, satisfying $(g - \mathring{g}) = o(r^{-1/2})$, $\partial g = o(r^{-3/2})$ and $\partial^2 g = o(r^{-5/2})$, is called an *asymptotically flat manifold* with N ends. We write $E_R = \{x \in \mathcal{M} \setminus K : r(x) > R\}$ to denote some exterior region, and define the set $B_R = \mathcal{M} \setminus \overline{E_R}$, which acts as a large ball in $\mathcal{M}$. We also make use of annular regions, $A_R = B_{2R} \setminus \overline{B_R}$. Note that the regions E_R and A_R are each comprised of N disconnected components.

Let $\mathfrak{g}$ be the Lie algebra of a compact Lie group, and let (A, ε) be initial data for the Yang-Mills fields as described in Section 2.2. Again, let γ be a positive definite inner product on $\mathfrak{g}$, with which Lie algebra indices are raised and lowered. By setting $\mathfrak{g} = \mathfrak{u}(1)$ it is clear that the results throughout this work are also valid for the Einstein-Maxwell system.

The regularity assumptions, $(g, A, \pi, \varepsilon) \in W^{2,2}_{loc} \times W^{2,2}_{loc} \times W^{1,2}_{loc} \times W^{1,2}_{loc}$, mentioned at the end of Section 2.2, are motivated by the following considerations. Two derivatives of the metric are required to make sense of the curvature, and since π is in some sense a derivative of the 4-metric, we expect to need one derivative. This is also required to make sense of the momentum constraint. Similarly, we need one derivative of the field strength tensor to make sense of the Yang-Mills equations, which amounts to taking two derivatives of A and one derivative of ε.

The phase space is then a Hilbert manifold modelled on the weighted Lebesgue and Sobolev spaces of Chapter 3, where the norms are defined with respect to $\mathring{g}$; that is,

$$\|u\|_{p,\delta} = \begin{cases} \left(\int_{\mathcal{M}} |u|^p \, r^{-\delta p - n} \sqrt{\mathring{g}} dx^n\right)^{1/p}, & p < \infty \\ \operatorname*{ess\,sup}_{\mathcal{M}} (r^{-\delta}|u|), & p = \infty \end{cases} \tag{4.1}$$

$$\|u\|_{k,p,\delta} = \sum_{|\alpha|=0}^{k} \|\mathring{\nabla}^{\alpha} u\|_{p,\delta-|\alpha|}. \tag{4.2}$$

For the study of asymptotically flat manifolds, it is sensible to have $(g - \mathring{g}) \in W^{2,2}_{-1/2}$, and therefore $\pi \in W^{1,2}_{-3/2}$. Imposing $\varepsilon \in W^{1,2}_{-3/2}$ enforces the usual $\frac{1}{r^2}$ fall off of the electric field in electromagnetism, however the appropriate domain for A is less obvious. Split $\mathfrak{g}$ into its centre, $\mathfrak{z}$, and a γ-orthogonal subspace, $\mathfrak{k}$, and decompose A into $A = A_{\mathfrak{z}} + A_{\mathfrak{k}}$, with $A_{\mathfrak{z}}$ valued in $\mathfrak{z}$ and $A_{\mathfrak{k}}$ valued in $\mathfrak{k}$. We consider A to be such that $A_{\mathfrak{z}} \in W^{2,2}_{-1/2}$ and $A_{\mathfrak{k}} \in W^{2,2}_{-3/2}$.

The decay conditions on A are chosen so that the gauge covariant derivative, $\hat{D} := \partial + [A, \cdot] \sim \partial + A_{\mathfrak{k}}$, behaves analogously to the usual covariant derivative at infinity; that is, $\hat{D}\theta = \partial\theta + o(r^{-3/2})\theta$. Although it may appear somewhat unnatural to require this condition for the analysis, such a condition is in fact required to ensure that the total charge is well-defined [25][1]. This condition also puts the electric and magnetic fields on equal footing (see Proposition 4.3). In the language of physics, we require that the Yang-Mills fields are asymptotic to photon fields before vanishing.

We are now ready to define the following spaces, which formally describe the phase space:

$$\mathcal{G} := W^{2,2}_{-1/2}(S_2), \qquad \mathcal{K} := W^{1,2}_{-3/2}(S^2 \otimes \Lambda^3),$$

$$\mathcal{A} := W^{2,2}_{-1/2}(T^*\mathcal{M} \otimes \mathfrak{z}) \oplus W^{2,2}_{-3/2}(T^*\mathcal{M} \otimes \mathfrak{k}), \qquad \mathcal{E} := W^{1,2}_{-3/2}(T\mathcal{M} \otimes \mathfrak{g}^* \otimes \Lambda^3),$$

[1] See also, the brief discussion under (5.11).

where S_2 and S^2 are symmetric covariant and contravariant 2-tensors, and Λ^3 and Λ^0 are volume forms and scalar functions on $\mathcal{M}$, respectively. The direct sum in the definition of $\mathcal{A}$ is to be understood as the internal direct sum in $W^{2,2}_{-1/2}(T^*\mathcal{M}\otimes\mathfrak{g})$, and the norm on $\mathcal{A}$ is given by the usual direct sum norm,

$$\|A\|_{\mathcal{A}} := \|A_{\mathfrak{z}}\|_{2,2,-1/2} + \|A_{\mathfrak{k}}\|_{2,2,-3/2}. \tag{4.3}$$

Further, define the spaces

$$\mathcal{G}^+ : = \{g \in S_2 \mid (g-\mathring{g}) \in \mathcal{G}, \quad g>0\}, \quad \mathcal{G}^+_\lambda : = \{g\in\mathcal{G}^+ \mid \lambda\mathring{g} < g < \lambda^{-1}\mathring{g}\},$$
$$\mathcal{N} : = L^2_{-1/2}(\Lambda^0 \times T\mathcal{M}\times\mathfrak{g}\otimes\Lambda^0), \qquad \mathcal{N}^* : = L^2_{-5/2}(\Lambda^3\times T^*\mathcal{M}\otimes\Lambda^3\times\mathfrak{g}^*\otimes\Lambda^3),$$

for $\lambda > 0$. The target space, $\mathcal{N}^*$, is the space of source terms, and its adjoint, $\mathcal{N}$, is identified with a space vector fields on the bundle, P.

The phase space for the Einstein-Yang-Mills equations is then

$$\mathcal{F} := \mathcal{G}^+ \times \mathcal{A}\times\mathcal{K}\times\mathcal{E}, \tag{4.4}$$

which does not depend on the choice of $\mathring{g}$ [11].

It is useful to note that both g and A are continuous by the Sobolev-Morrey embedding, $C^{0,1/2} \subset W^{2,2}$ (see, for example, [37]), and therefore the conditions defining $\mathcal{G}^+$ and $\mathcal{G}^+_\lambda$ are understood in the pointwise sense. In particular, if $g\in\mathcal{G}^+_\lambda$ then

$$\lambda\mathring{g}_{ij}(x)v^iv^j < g_{ij}(x)v^iv^j < \lambda^{-1}\mathring{g}_{ij}(x)v^iv^j \tag{4.5}$$

for all $x\in\mathcal{M}$ and $v\in T_x\mathcal{M}$.

4.2 Elementary Estimates

For the sake of presentation, some elementary estimates are gathered here to be referenced in subsequent sections. Throughout this thesis, c is used to denote arbitrary constants, which may change from line to line. The first estimate below is very simple, however as it is used frequently throughout, it is included as a stand-alone Proposition.

Proposition 4.1. *For $u \in W^{1,2}_{\delta}$,*

$$\|u^2\|_{2,2\delta} \leq c\|u\|^2_{1,2,\delta}. \tag{4.6}$$

Proof. Simply noting that $\|u^2\|_{2,2\delta} = \left(\int_{\mathcal{M}} |u|^4 r^{-4\delta-n} dx^n\right)^{1/2} = \|u\|^2_{4,\delta}$, the weighted Sobolev inequality (3.6) gives (4.6). □

Since the Riemann curvature behaves like a second derivative of the metric and the scalar curvature is the dominant term in the Hamiltonian constraint (2.2), the following $L^2_{-5/2}$ estimates are required.

Proposition 4.2. *For $g \in \mathcal{G}^+_{\lambda}$,*

$$\|R(g)\|_{2,-5/2} \leq c(\lambda)\|Ric(g)\|_{2,-5/2} \leq c(\lambda)\|Riem(g)\|_{2,-5/2} \leq c(\lambda)(1+\|\mathring{\nabla} g\|^2_{1,2,-3/2}),$$

where $c(\lambda)$ is some constant depending on λ, which may vary between each inequality.

Proof. The Riemann curvature may be expressed symbolically as

$$Riem \sim \mathring{Riem} + g^{-1}\mathring{\nabla}^2 g + (g^{-1})^2(\mathring{\nabla} g)^2,$$

where $\mathring{Riem}$ refers to the Riemann curvature of $\mathring{g}$. See Appendix A for the full expression. Making use of (3.6), (4.5) and (4.6),

$$\begin{aligned}
\|Riem(g)\|_{2,-5/2} &\le c(\lambda)(1 + \|\mathring{\nabla}^2 g\|_{2,-5/2} + \|(\mathring{\nabla} g)^2\|_{2,-5/2}) \\
&\le c(\lambda)(1 + \|\mathring{\nabla} g\|_{1,2,-3/2} + \|\mathring{\nabla} g\|^2_{1,2,-3/2})\| \\
&\le c(\lambda)(1 + \|\mathring{\nabla} g\|^2_{1,2,-3/2}).
\end{aligned}$$

Since the Ricci and scalar curvatures are simply obtained by taking traces, their norms are clearly bound by $c(\lambda)\|Riem(g)\|$. □

The following Proposition shows that the decay conditions on A ensure that the electric and magnetic fields are considered on the same footing.

Proposition 4.3. *If $A \in \mathcal{A}$, then*

$$\|B\|_{1,2,-3/2} \le c\left(1 + \|A\|^2_{\mathcal{A}}\right), \tag{4.7}$$

and in particular, $B \in \mathcal{E}$.

Proof. Making use of (3.4) and (3.6),

$$\begin{aligned}
\|\mathring{\nabla} B\|_{2,-5/2} &\le c\left(\|\mathring{\nabla}^2 A\|_{2,-5/2} + \|\mathring{\nabla}(A_{\mathfrak{k}})A_{\mathfrak{k}}\|_{2,-5/2}\right) \\
&\le c\left(\|\mathring{\nabla}^2 A\|_{2,-5/2} + \|\mathring{\nabla} A_{\mathfrak{k}}\|_{2,-5/2}\|A_{\mathfrak{k}}\|_{\infty,0}\right) \\
&\le c\left(\|\mathring{\nabla}^2 A\|_{2,-5/2} + \|\mathring{\nabla} A_{\mathfrak{k}}\|_{2,-5/2}\|A_{\mathfrak{k}}\|_{2,2,0}\right) \\
&\le c\left(1 + \|A\|^2_{\mathcal{A}}\right).
\end{aligned}$$

The proof is completed by an application of the weighted Poincaré inequality (3.8). □

The difference of Christoffel symbols tensor, $\tilde{\Gamma} := \Gamma - \mathring{\Gamma}$, is also useful for several estimates. This can be controlled by the norm of g as follows.

Proposition 4.4. *For $g \in \mathcal{G}_\lambda^+$, $\lambda > 0$ we have*

$$\|\tilde{\Gamma}\|_{1,2,-3/2} \leq c(\lambda)(1 + \|\mathring{\nabla} g\|^2_{1,2,-3/2}). \tag{4.8}$$

Proof. Employing the standard trick of fixing a point $p \in \mathcal{M}$ and choosing coordinates such that $\mathring{\Gamma} = 0$ and $\mathring{\nabla} = \partial$ at p, we have

$$\tilde{\Gamma}^i_{jk} = \frac{1}{2} g^{il}(\mathring{\nabla}_j g_{lk} + \mathring{\nabla}_k g_{jl} - \mathring{\nabla}_l g_{jk}) \tag{4.9}$$

at p. Since (4.9) is tensorial, its validity is independent of the coordinate choice and as p was arbitrary the expression holds everywhere on $\mathcal{M}$. Differentiating (4.9), and making use of (4.5) and (4.6) yields

$$\|\mathring{\nabla}\tilde{\Gamma}\|_{2,-5/2} \leq c(\lambda) \left(\|(\mathring{\nabla} g)^2\|_{2,-5/2} + \|\mathring{\nabla}^2 g\|_{2,-5/2} \right) \tag{4.10}$$

$$\leq c(\lambda) \left(\|\mathring{\nabla} g\|^2_{1,2,-3/2} + \|\mathring{\nabla}^2 g\|_{2,-5/2} \right) \tag{4.11}$$

$$\leq c(\lambda)(1 + \|\mathring{\nabla} g\|^2_{1,2,-3/2}). \tag{4.12}$$

An application of the weighted Poincaré inequality (3.8) now completes the proof. □

4.3 The Constraint Map

We begin by writing the left-hand side of the constraint equations, (2.2), (2.3), (2.14), as a map acting on the phase space:

$$\begin{aligned}\Phi_0(g, A, \pi, \varepsilon) &= \left(\frac{1}{2}(\pi^k_k)^2 - \pi^{ij}\pi_{ij} - (\frac{1}{8}\varepsilon^k_a \varepsilon^a_k + 2B^k_a B^a_k) \right) g^{-1/2} + R\sqrt{g} \\ &= \left(\frac{1}{2}(\pi^k_k)^2 - \pi^{ij}\pi_{ij} - 2(E^k_a E^a_k + B^k_a B^a_k) \right) g^{-1/2} + R\sqrt{g},\end{aligned} \tag{4.13}$$

$$\Phi_i(g, A, \pi, \varepsilon) = 2\nabla^j \pi_{ij} - \varepsilon^j_a(\mathring{\nabla}_i A^a_j - \mathring{\nabla}_j A^a_i) + A^a_i \mathring{\nabla}_j \varepsilon^j_a, \tag{4.14}$$

$$\Phi_a(g, A, \pi, \varepsilon) = -\mathring{\nabla}_j \varepsilon^j_a - C^c_{ab} A^b_j \varepsilon^j_c, \tag{4.15}$$

where the Yang-Mills stress-energy tensor is included in the definitions of Φ_0 and Φ_i. This is called the constraint map and level sets of Φ correspond to the sets of solutions to the Einstein-Yang-Mills constraints, for fixed sources. Note that $\mathring{\nabla}$ is used in place of ∇ in (4.14) and (4.15). This is due to the antisymmetry in the derivatives of A, and since ε is a vector density; any torsion-free connection may be used here. The momentum constraint sometimes differs from the one used here by the term, $A_i^a(\mathring{\nabla}_j\varepsilon_a^j + C_{ab}^c A_j^b \varepsilon_c^j)$, however this is a simply a multiple of the Gauss constraint so adopting the alternative momentum constraint makes no difference to the conclusions.

As it is to be shown that the level sets of the constraint map are smooth submanifolds of $\mathcal{F}$, it is required that we show that this map is indeed smooth. For this, we first establish the following estimate:

Proposition 4.5. *Suppose* $(g, A, \pi, \varepsilon) \in \mathcal{G}_\lambda^+ \times \mathcal{A} \times \mathcal{K} \times \mathcal{E} \subset \mathcal{F}$ *for some fixed* $\lambda > 0$, *then there exists a constant* $c(\lambda)$ *such that*

$$\begin{aligned}
\|\Phi_0(g, A, \pi, \varepsilon)\|_{2,-5/2} &\leq c(\lambda)(1 + \|g - \mathring{g}\|^2_{2,2,-1/2} + \|\pi\|^2_{1,2,-3/2} \\
&\qquad + \|\varepsilon\|^2_{1,2,-3/2} + \|A\|^4_{\mathcal{A}}), && (4.16)\\
\|\Phi_i(g, A, \pi, \varepsilon)\|_{2,-5/2} &\leq c(\lambda)\big(\|\pi\|_{1,2,-3/2}(1 + \|\mathring{\nabla} g\|^2_{1,2,-3/2}) \\
&\qquad + \|\varepsilon\|_{1,2,-3/2}\|A\|_{\mathcal{A}}\big), && (4.17)\\
\|\Phi_a(g, A, \pi, \varepsilon)\|_{2,-5/2} &\leq c(\lambda)\, \|\varepsilon\|_{1,2,-3/2}\, (1 + \|A\|_{\mathcal{A}}). && (4.18)
\end{aligned}$$

Proof. Making use of the pointwise bounds (4.5), and Propositions 4.1, 4.2 and 4.3,

$$\begin{aligned}
\|\Phi_0(g, A, \pi, \varepsilon)\|_{2,-5/2} &\leq c(\lambda)\big(\|R\|_{2,-5/2} + \|\pi^2\|_{2,-5/2} + \|\varepsilon^2\|_{2,-5/2} + \|B^2\|_{2,-5/2}\big) \\
&\leq c(\lambda)\big(1 + \|\mathring{\nabla} g\|^2_{1,2,-3/2} + \|\pi\|^2_{1,2,-3/2} + \|\varepsilon\|^2_{1,2,-3/2} \\
&\qquad + \|B\|^2_{1,2,-3/2}\big) \\
&\leq c(\lambda)\big(1 + \|\mathring{\nabla} g\|^2_{1,2,-3/2} + \|\pi\|^2_{1,2,-3/2} + \|\varepsilon\|^2_{1,2,-3/2} + \|A\|^4_{\mathcal{A}}\big),
\end{aligned}$$

which establishes (4.16).

Using the weighted Sobolev (3.6) and Hölder (3.4) inequalities again, (4.17) is similarly obtained:

$$\begin{aligned}
\|\Phi_i(g,A,\pi,\varepsilon)\|_{2,-5/2} &\le c\left(\|\nabla\pi\|_{2,-5/2} + \|\varepsilon\mathring{\nabla}A\|_{2,-5/2} + \|A\mathring{\nabla}\varepsilon\|_{2,-5/2}\right)\\
&\le c\left(\|\mathring{\nabla}\pi\|_{2,-5/2} + \|\tilde{\Gamma}\pi\|_{2,-5/2} + \|\varepsilon\|_{4,-5/4}\|\mathring{\nabla}A\|_{4,-5/4}\right.\\
&\quad \left. + \|A\|_{\infty,0}\|\mathring{\nabla}\varepsilon\|_{2,-5/2}\right)\\
&\le c\left(\|\pi\|_{1,2,-3/2} + \|\tilde{\Gamma}\|_{4,-5/4}\|\pi\|_{4,-5/4} + \|\varepsilon\|_{1,2,-3/2}\|A\|_{2,2,-1/2}\right)\\
&\le c\left(\|\pi\|_{1,2,-3/2}(1+\|\mathring{\nabla}g\|^2_{1,2,-3/2}) + \|\varepsilon\|_{1,2,-3/2}\|A\|_{\mathcal{A}}\right).
\end{aligned}$$

Finally,

$$\begin{aligned}
\|\Phi_a(g,A,\pi,\varepsilon)\|_{2,-5/2} &\le c(\|\varepsilon\|_{1,2,-3/2} + \|A_{\mathfrak{k}}\|_{4,-5/4}\|\varepsilon\|_{4,-5/4})\\
&\le c(\|\varepsilon\|_{1,2,-3/2} + \|A_{\mathfrak{k}}\|_{1,2,-3/2}\|\varepsilon\|_{1,2,-3/2})\\
&\le c\|\varepsilon\|_{1,2,-3/2}(1+\|A\|_{\mathcal{A}}),
\end{aligned}$$

establishing (4.18) and completing the proof. □

From this estimate it can be shown that Φ is a smooth map.

Theorem 4.6. $\Phi : \mathcal{F} \to \mathcal{N}^*$ *is a smooth map of Hilbert manifolds.*

Proof. It can be seen from Proposition 4.5, that $\Phi : \mathcal{F} \to \mathcal{N}^*$ is locally bounded. The scalar curvature can be expressed as a polynomial function in g, g^{-1}, $\mathring{\nabla}g$ and $\mathring{\nabla}^2 g$ (cf. Appendix A), and therefore the constraint map can be expressed as a polynomial function in 12 variables,

$$\tilde{\Phi}(g, g^{-1}, \sqrt{g}, 1/\sqrt{g}, \mathring{\nabla}g, \mathring{\nabla}^2 g, \pi, \mathring{\nabla}\pi, \varepsilon, \mathring{\nabla}\varepsilon, A, \mathring{\nabla}A) = \Phi(g,A,\pi,\varepsilon). \qquad (4.19)$$

For positive definite matrices, the maps $g \mapsto \mathring{\nabla}g$, $g \mapsto \mathring{\nabla}^2 g$, $A \mapsto \mathring{\nabla}A$, $g \mapsto \sqrt{g}$, etc. are smooth. Further, locally bounded polynomial functions are smooth (in the sense of Fréchet differentiability) (see, for example, [41], Chapter 26). It follows that Φ is a smooth map of Hilbert manifolds. $\square$

In the above we restricted ourselves to G_λ^+ to obtain the local bound, however this is not necessary. We could instead express each g as $(g - \mathring{g}) + \mathring{g}$ and an estimate independent of λ is obtained, however the use of λ simplifies the estimates significantly.

4.4 The Operators $D\Phi$ and $D\Phi^*$

The following consequence of the implicit function theorem (see [27], Chapter VII) is central to establishing that the level sets of Φ are Hilbert submanifolds of $\mathcal{F}$.

Theorem 4.7. *Let $f : X \to Y$ be a C^1 mapping between Banach manifolds. If $Df_x : T_xX \to T_{f(x)}Y$ is surjective and has complementable kernel for all $x \in S := f^{-1}(c)$, for some $c \in Y$, then S is a submanifold of X.*

In the above, complementable kernel means that the T_xX splits into the direct sum of $\ker Df_x$ and a closed complementary subspace.

Theorem 4.7 motivates an examination of the linearised constraint map, $D\Phi_{(g,A,\pi,\varepsilon)} : T_{(g,A,\pi,\varepsilon)}\mathcal{F} \to \mathcal{N}^*$, and its formal adjoint, $D\Phi^*_{(g,A,\pi,\varepsilon)}$. In this section we establish some results concerning $D\Phi_{(g,A,\pi,\varepsilon)}$ and $D\Phi^*_{(g,A,\pi,\varepsilon)}$, and we return to Theorem 4.7 in Section 4.5.

Where there is no risk of confusion, reference to the base point, (g, A, π, ε), is omitted and we simply write $D\Phi$ or $D\Phi^*$. For convenience, we use the notation $D\Phi_{(g,A,\pi,\varepsilon)} = (D\Phi_0, D\Phi_i, D\Phi_a)$ and $D\Phi^*_{(g,A,\pi,\varepsilon)} = (D\Phi^*_g, D\Phi^*_A, D\Phi^*_\pi, D\Phi^*_\varepsilon)$. While cumbersome to explicitly write, it is a simple computation to calculate the operators $D\Phi$ and $D\Phi^*$ (cf. [2, 33]). The formal adjoint is computed by acting $D\Phi$ on a variation $(h, b, p, f) \in T_{(g,A,\pi,\varepsilon)}\mathcal{F}$, pairing it with $(N, X, V) \in \mathcal{N}$, and formally integrating by parts. We arrive at the expressions,

$$\begin{aligned} D\Phi_0[h,b,p,f] = &(\pi^k_k \pi^{ij} - 2\pi^{ik}\pi^j_k - 2(E^i_a E^{aj} + B^i_a B^{aj}))h_{ij} g^{-1/2} \\ &+ (\frac{1}{2}\pi^{ij}\pi_{ij} - \frac{1}{4}(\pi^k_k)^2 + (E^k_a E^a_k + B^k_a B^a_k))h^j_j g^{-1/2} \\ &+ (\frac{1}{2}h^k_k R - \Delta h^k_k + \nabla^i\nabla^j h_{ij} - R^{ij}h_{ij})\sqrt{g} \\ &- 4\epsilon^{ijk}(\mathring{\nabla}_j b^a_k + C^a_{bc}A^b_j b^c_k)B_{ai} g^{-1/2} \\ &+ (p^k_k \pi^j_j - 2\pi^{ij}p_{ij})g^{-1/2} + f^i_a E^a_i g^{-1/2}, \end{aligned} \tag{4.20}$$

$$\begin{aligned} D\Phi_i[h,b,p,f] = &2\nabla_j(\pi^{jk}h_{ik}) - \pi^{jk}\nabla_i h_{jk} - \varepsilon^j_a(\mathring{\nabla}_i b^a_j - \mathring{\nabla}_j b^a_i) + b^a_i \mathring{\nabla}_j \varepsilon^j_a \\ &+ 2\nabla_j p^j_i - f^j_a(\mathring{\nabla}_i A^a_j - \mathring{\nabla}_j A^a_i) + A^a_i \mathring{\nabla}_j f^j_a, \end{aligned} \tag{4.21}$$

$$D\Phi_a[h,b,p,f] = -(C^c_{ab}\varepsilon^j_c b^b_j + \mathring{\nabla}_j f^j_a + C^c_{ab} f^j_c A^b_j), \tag{4.22}$$

and

$$\begin{aligned} D\Phi^*_g[N,X,V] = &\mathcal{L}_X \pi^{ij} + N\Big(\pi^k_k\pi^{ij} - 2\pi^{ik}\pi^j_k - 2(E^i_a E^{aj} + B^i_a B^{ai}) \\ &+ \left\{\frac{1}{2}\pi^{kl}\pi_{kl} - \frac{1}{4}(\pi^k_k)^2 + (E^k_a E^a_k + B^k_a B^a_k)\right\} g^{ij}\Big) g^{-1/2} \\ &+ \left\{N(\frac{1}{2}Rg^{ij} - R^{ij}) + \nabla^i\nabla^j N - g^{ij}\nabla^k\nabla_k N\right\}\sqrt{g}, \end{aligned} \tag{4.23}$$

$$\begin{aligned} D\Phi^*_A[N,X,V] = &\mathcal{L}_X \varepsilon^i_a - C^b_{ca}\varepsilon^i_b V^c \\ &- 4\epsilon^{ijk}\left\{\nabla_j(NB_{ak}g^{-1/2}) + C_{abc}NB^c_k A^b_j g^{-1/2}\right\}, \end{aligned} \tag{4.24}$$

$$D\Phi^*_\pi[N,X,V] = N(g_{ij}\pi^k_k - 2\pi_{ij})g^{-1/2} - \mathcal{L}_X g_{ij}, \tag{4.25}$$

$$D\Phi^*_\varepsilon[N,X,V] = -\frac{1}{4}N\varepsilon^a_i g^{-1/2} - \mathcal{L}_X A^a_i + \mathring{\nabla}_i V^a + C^a_{bc}A^b_i V^c. \tag{4.26}$$

In the above, $\mathcal{L}$ denotes the usual Lie derivative, which can be expressed in term of a given connection as follows:

$$\begin{aligned}
\mathcal{L}_X g_{ij} &= X^k \mathring{\nabla}_k g_{ij} + g_{kj} \mathring{\nabla}_i X^k + g_{ki} \mathring{\nabla}_j X^k \\
&= \nabla_i X_j + \nabla_j X_i \\
\mathcal{L}_X A^a_i &= X^k \mathring{\nabla}_k A^a_i + A^a_k \mathring{\nabla}_i X^k \\
\mathcal{L}_X \pi^{ij} &= X^k \mathring{\nabla}_k \pi^{ij} - \pi^{jk} \mathring{\nabla}_k X^i - \pi^{ik} \mathring{\nabla}_k X^j + \pi^{ij} \mathring{\nabla}_k X^k \\
\mathcal{L}_X \varepsilon^i_a &= X^k \mathring{\nabla}_k \varepsilon^i_a - \varepsilon^k_a \mathring{\nabla}_k X^i + \varepsilon^i_a \mathring{\nabla}_k X^k.
\end{aligned}$$

The expressions (4.23)-(4.26) are in terms of the Lie derivative, rather than the connection, to illuminate their relationship with the evolution equations, (2.4), (2.5), (2.15), (2.16). We write $\xi = (\xi^0, \xi^i, \xi^a) = (N, X, V) \in \mathcal{N}$, which is to be identified with a vector field on the principal bundle.

We first establish that the kernel of $D\Phi^*$ is trivial. In the cases considered by Fischer, Marsden [32, 33] and Arms [1, 2], where $\mathcal{M}$ is a compact manifold, this is not necessarily true. For some particular solutions, (g, A, π, ε), the kernel of $D\Phi^*_{(g,A,\pi,\varepsilon)}$ does indeed contain non-trivial elements, and these solutions are exactly those that are not linearisation stable [48]. Non-trivial elements of the kernel correspond to vector fields on the bundle whose integral curves are symmetries of the initial data [48, 49]. This is motivated by the fact,

$$\frac{\partial}{\partial t}(\pi, \varepsilon, -g, -A) = D\Phi^*[\xi],$$

where (N, X) is the lapse-shift 4-vector and V is the electric potential (cf. (2.4), (2.5), (2.15), (2.16)). This is discussed in greater detail in Chapter 6.

Note, the expressions (4.23)-(4.26) are only understood for suitably differentiable ξ, however $D\Phi^*$ acts on a space with only L^2 regularity. That is, $D\Phi^*[\xi]$ may only be understood in the weak sense for a generic $\xi \in \mathcal{N}$. A weak solution of $D\Phi^*[\xi] = f$ is an element $\xi \in \mathcal{N}$ such that

$$\int_{\mathcal{M}} \xi \cdot D\Phi[h, b, p, f] = \int_{\mathcal{M}} f \cdot (h, b, p, f), \tag{4.27}$$

for all $(h, b, p, f) \in \mathcal{G} \times \mathcal{A} \times \mathcal{K} \times \mathcal{E} = T_{(g,A,\pi,\varepsilon)}\mathcal{F}$.

In order to examine the kernel of $D\Phi^*$, it is first shown that a weak solution of $D\Phi^*[\xi] = (f_1, f_2, f_3, f_4)$, where $(f_1, f_3, f_4) \in L^2_{-5/2} \times W^{1,2}_{-3/2} \times W^{1,2}_{-3/2}$, is in fact a strong solution. To demonstrate this, we require the coercivity estimate given by Lemma 4.8 below. Note that in the proof of this estimate, and throughout the rest of the thesis, C denotes an arbitrary constant that depends on a fixed, $(g, A, \pi, \varepsilon) \in \mathcal{F}$. When the constant changes from c to C, this is to indicate that the constant has absorbed terms depending on (g, A, π, ε).

Lemma 4.8. *If* $\xi \in W^{2,2}_{-1/2}$ *satisfies* $D\Phi^*[\xi] = (f_1, f_2, f_3, f_4)$ *with* $(f_1, f_3, f_4) \in L^2_{-5/2} \times W^{1,2}_{-3/2} \times W^{1,2}_{-3/2}$, *then*

$$\|\xi\|_{2,2,-1/2} \leq C\Big(\|f_1\|_{2,-5/2} + \|(f_3, f_4)\|_{1,2,-3/2} + \|\xi\|_{2,0}\Big). \tag{4.28}$$

Proof. The weighted Poincaré inequality (3.8) gives

$$\|\xi\|_{2,2,-1/2} \leq c\|\mathring{\nabla}^2\xi\|_{2,-5/2}, \tag{4.29}$$

and therefore only an estimate for the second derivative is required.

Rearranging (4.23) gives

$$\nabla^i\nabla^j N - g^{ij}\nabla^k\nabla_k N = S^{ij},$$

where S is given by

$$\sqrt{g}S^{ij} = D\Phi_g^*[\xi]^{ij} - N\Big(\pi_k^k\pi^{ij} - 2\pi^{ik}\pi_k^j - 2(E_a^iE^{aj} + B_a^iB^{ai}) + \left\{\frac{1}{2}\pi^{kl}\pi_{kl} - \frac{1}{4}(\pi_k^k)^2 + (E_a^kE_k^a + B_a^kB_k^a)\right\} g^{ij}\Big)g^{-1/2} - \left\{N(\frac{1}{2}Rg^{ij} - R^{ij})\right\}\sqrt{g} + \mathcal{L}_X\pi^{ij}.$$

From this, we can then write

$$\nabla^i\nabla^jN = S^{ij} - \frac{1}{2}g^{ij}S_k^k, \tag{4.30}$$

which gives an estimate for $\nabla^2 N$:

$$\|\nabla^2N\|_{2,-5/2} \leq C\|S\|_{2,-5/2}. \tag{4.31}$$

The standard weighted Sobolev-type inequalities are applied again to give

$$\begin{aligned}
\|\mathring{\nabla}^2N\|_{2,-5/2} \leq & c\Big(\|D\Phi_g^*[\xi]\|_{2,-5.2} + \|\tilde{\Gamma}\mathring{\nabla}N\|_{2,-5/2} + \|\pi\mathring{\nabla}X\|_{2,-5/2} \\
& + \|X\mathring{\nabla}\pi\|_{2,-5/2} + \|N\|_{\infty,0}(\|\pi^2\|_{2,-5/2} + \|E^2\|_{2,-5/2} \\
& + \|B^2\|_{2,-5/2} + \|Ric(g)\|_{2,-5/2})\Big) \\
\leq & C(\|f_1\|_{2,-5/2} + \|\xi\|_{\infty,0} + \|\mathring{\nabla}\xi\|_{3,-1}(\|\tilde{\Gamma}\|_{6,-3/2} + \|\pi\|_{6,-3/2})) \\
\leq & C(\|f_1\|_{2,-5/2} + \|\xi\|_{\infty,0} + \|\mathring{\nabla}\xi\|_{3,-1}(\|\tilde{\Gamma}\|_{1,2,-3/2} + \|\pi\|_{1,2,-3/2})) \\
\leq & C(\|f_1\|_{2,-5/2} + \|\xi\|_{\infty,0} + \|\mathring{\nabla}\xi\|_{3,-1}).
\end{aligned}$$

By making use of the Riemannian curvature identity,

$$R_{ijkl}X^l = \nabla_i\nabla_jX_k - \nabla_j\nabla_iX_k, \tag{4.32}$$

$\nabla^2 X$ may be expressed in terms of $D\Phi_\pi^*[\xi]$. To see this, note that the Bianchi identity gives the following:

$$
\begin{aligned}
\nabla_k \mathcal{L}_X g_{ij} + \nabla_j \mathcal{L}_X g_{ik} - \nabla_i \mathcal{L}_X g_{jk} =& \nabla_j \nabla_k X_i + \nabla_i \nabla_k X_j + \nabla_k \nabla_j X_i \\
&+ \nabla_i \nabla_j X_k - \nabla_k \nabla_i X_j - \nabla_j \nabla_i X_k \\
=& \nabla_j \nabla_k X_i + R_{ikjl} X^l + \nabla_k \nabla_j X_i \\
&+ R_{ijkl} X^l + (\nabla_k \nabla_j X_i - \nabla_k \nabla_j X_i) \\
=& R_{jkil} X^l + R_{ikjl} X^l + R_{ijkl} X^l + 2\nabla_k \nabla_j X_i \\
=& 2(R_{ikjl} X^l + \nabla_k \nabla_j X_i),
\end{aligned}
$$

which relates $\mathcal{L}_X g$ to $\nabla^2 X$. In particular, we have

$$
\|\nabla^2 X\|_{2,-5/2} \leq c(\|Riem(g)\|_{2,-5/2} \|X\|_{\infty,0} + \|\nabla \mathcal{L}_X g\|_{2,-5/2}). \tag{4.33}
$$

Making use of (4.14), the Lie derivative is expressed as

$$
\mathcal{L}_X g_{ij} = N(g_{ij}\pi_k^k - 2\pi_{ij}) g^{-1/2} - (D\Phi_\pi^*[\xi])_{ij}, \tag{4.34}
$$

and the weighted Sobolev-type inequalities give

$$
\begin{aligned}
\|\nabla \mathcal{L}_X g\|_{2,-5/2} \leq& \, c\,(\|\nabla(N\pi)\|_{2,-5/2} + \|\nabla f_3\|_{2,-5/2}) \\
\leq& \, c\Big(\|\mathring{\nabla} f_3\|_{2,-5/2} + \|\tilde{\Gamma}\|_{4,-1} \|f_3\|_{4,-3/2} + \|\mathring{\nabla} N\|_{3,-1} \|\pi\|_{6,-3/2} \\
&+ \|N\|_{\infty,0} (\|\mathring{\nabla}\pi\|_{2,-5/2} + \|\tilde{\Gamma}\pi\|_{2,-5/2})\Big) \\
\leq& \, c\Big(\|\mathring{\nabla} f_3\|_{2,-5/2} + \|\mathring{\nabla} g\|_{2,2,-1/2} \|f_3\|_{1,2,-3/2} + \|\mathring{\nabla} N\|_{3,-1} \|\pi\|_{1,2,-3/2} \\
&+ \|N\|_{\infty,0} (\|\mathring{\nabla}\pi\|_{2,-5/2} + \|\mathring{\nabla} g\|_{2,2.-1/2} \|\pi\|_{1,2,-3/2})\Big) \\
\leq& \, C\left(\|f_3\|_{1,2,-3/2} + \|N\|_{\infty,0} + \|\mathring{\nabla} N\|_{3,-1}\right).
\end{aligned}
$$

We now obtain an estimate for $\|\mathring{\nabla}^2 X\|$ in terms of $\|\nabla^2 X\|$ as follows:

$$\begin{aligned}
\|\mathring{\nabla}^2 X\|_{2,-5/2} \leq c\,\big(&\|\nabla^2 X\|_{2,-5/2} + \|\mathring{\nabla}(X)\tilde{\Gamma}\|_{2,-5/2} + \|X\mathring{\nabla}(\tilde{\Gamma})\|_{2,-5/2} \\
&+ \|\tilde{\Gamma}^2 X\|_{2,-5/2}\big) \\
\leq c\,\big(&\|\nabla^2 X\|_{2,-5/2} + \|\mathring{\nabla} X\|_{3,-1}\|\tilde{\Gamma}\|_{6,-3/2} \\
&+ \|X\|_{\infty,0}(\|\mathring{\nabla}\tilde{\Gamma}\|_{2,-5/2} + \|\tilde{\Gamma}^2\|_{2,-5/2})\big) \\
\leq c\,\big(&\|\nabla^2 X\|_{2,-5/2} + \|\mathring{\nabla} X\|_{3,-1}\|\tilde{\Gamma}\|_{1,2,-3/2} \\
&+ \|X\|_{\infty,0}(\|\mathring{\nabla}\tilde{\Gamma}\|_{2,-5/2} + \|\tilde{\Gamma}\|^2_{1,2,-3/2})\big) \\
\leq C\,\big(&\|\nabla^2 X\|_{2,-5/2} + \|\mathring{\nabla} X\|_{3,-1} + \|X\|_{\infty,0}\big).
\end{aligned}$$

Putting all of this together gives the estimate,

$$\|\mathring{\nabla}^2 X\|_{2,-5/2} \leq C(\|f_3\|_{1,2,-3/2} + \|\xi\|_{\infty,0} + \|\mathring{\nabla}\xi\|_{3,-1}).$$

Similarly, from (4.26) we have

$$\mathring{\nabla}_i V^a = f^a_{4i} + \frac{1}{4} N \varepsilon^a_i g^{-1/2} + X^k \mathring{\nabla}_k A^a_i + A^a_k \mathring{\nabla}_i X^k - C^a_{bc} A^b_i V^c, \tag{4.35}$$

which can be differentiated to obtain an estimate for $\mathring{\nabla}^2 V$:

$$\begin{aligned}
\|\mathring{\nabla}^2 V\|_{2,-5/2} \leq c\Big(&\|f_4\|_{1,2,-3/2} + \|\varepsilon\mathring{\nabla} N\|_{2,-5/2} + \|N\mathring{\nabla}\varepsilon\|_{2,-5/2} \\
&+ \|A\mathring{\nabla}^2 X\|_{2,-5/2} + \|X\mathring{\nabla}^2 A\|_{2,-5/2} + \|\mathring{\nabla} X\mathring{\nabla} A\|_{2,-5/2} \\
&+ \|A_{\mathfrak{k}}\mathring{\nabla} V\|_{2,-5/2} + \|V\mathring{\nabla} A_{\mathfrak{k}}\|_{2,-5/2}\Big) \\
\leq c\Big(&\|f_4\|_{1,2,-3/2} + \|\mathring{\nabla}\xi\|_{3,-1}(\|\varepsilon\|_{6,-3/2} + \|\mathring{\nabla} A\|_{6,-3/2} \\
&+ \|A_{\mathfrak{k}}\|_{6,-3/2}) + \|\xi\|_{\infty,0}(\|\mathring{\nabla}\varepsilon\|_{2,-5/2} + \|\mathring{\nabla}^2 A\|_{2,-5/2} \\
&+ \|\mathring{\nabla} A_{\mathfrak{k}}\|_{2,-5/2}) + \|A\|_{\infty,-1/2}\|\mathring{\nabla}^2 X\|_{2,-2}\Big)
\end{aligned}$$

$$\begin{aligned}
\|\mathring{\nabla}^2 V\|_{2,-5/2} \leq & c\Big(\|f_4\|_{1,2,-3/2} + \|\mathring{\nabla}\xi\|_{3,-1}(\|\varepsilon\|_{1,2,-3/2} + \|A\|_{2,2,-3/2} \\
& + \|A_{\mathfrak{k}}\|_{1,2,-3/2}) + \|\xi\|_{\infty,0}(\|\varepsilon\|_{1,2,-3/2} + \|A\|_{\mathcal{A}}) \\
& + \|A\|_{2,2,-1/2}\|\mathring{\nabla}^2 X\|_{2,-2}\Big) \\
\leq & C(\|f_4\|_{1,2,-3/2} + \|f_3\|_{1,2,-3/2} + \|\xi\|_{\infty,0} + \|\mathring{\nabla}\xi\|_{3,-1}).
\end{aligned}$$

Note that we have made use of the estimate for $\|\mathring{\nabla}^2 X\|_{2,-5/2}$ above. We can now combine the separate estimates to obtain

$$\|\mathring{\nabla}^2\xi\|_{2,-5/2} \leq C(\|f_1\|_{2,-5/2} + \|(f_3, f_4)\|_{1,2,-3/2} + \|\xi\|_{\infty,0} + \|\mathring{\nabla}\xi\|_{3,-1}). \tag{4.36}$$

We can estimate the last two terms on the right-hand side using the weighted inequalities, Young's inequality, and the definition of the $W^{k,p}_\delta$ norm directly:

$$\begin{aligned}
\|\xi\|_{\infty,0} \leq & c\|\xi\|_{1,4,0} = \|\xi^{1/4}\xi^{3/4}\|_{1,4,0} \\
& \leq c\|\xi^{1/4}\|_{1,8,0}\|\xi^{3/4}\|_{1,8,0} \\
& \leq c\|\xi\|^{1/4}_{1,2,0}\|\xi\|^{3/4}_{1,6,0} \\
& \leq c\|\xi\|^{1/4}_{1,2,0}\|\xi\|^{3/4}_{2,2,0} \\
& \leq c\|\xi\|^{1/4}_{1,2,0}(\|\xi\|_{1,2,0} + \|\mathring{\nabla}^2\xi\|_{2,-2})^{3/4} \\
& \leq c\epsilon^{-3}\|\xi\|_{1,2,0} + \epsilon(\|\xi\|_{1,2,0} + \|\mathring{\nabla}^2\xi\|_{2,-2}) \\
& \leq c\epsilon^{-3}\|\xi\|_{1,2,0} + \epsilon\|\mathring{\nabla}^2\xi\|_{2,-2},
\end{aligned} \tag{4.37}$$

for any $\epsilon > 0$.

An estimate for the final term in (4.36) is obtained almost identically:

$$\begin{aligned}\|\mathring{\nabla}\xi\|_{3,-1} \leq& \|\xi\|_{1,3,0} = \|\xi^{1/3}\xi^{2/3}\|_{1,3,0} \\ &\leq c\|\xi^{1/3}\|_{1,6,0}\|\xi^{2/3}\|_{1,6,0} \\ &\leq c\|\xi\|_{1,2,0}^{1/3}\|\xi\|_{1,4,0}^{2/3} \\ &\leq c\|\xi\|_{1,2,0}^{1/3}\|\xi\|_{2,2,0}^{2/3} \\ &\leq c\|\xi\|_{1,2,0}^{1/3}(\|\xi\|_{1,2,0} + \|\mathring{\nabla}^2\xi\|_{2,-2})^{2/3} \\ &\leq c\epsilon^{-2}\|\xi\|_{1,2,0} + \epsilon(\|\xi\|_{1,2,0} + \|\mathring{\nabla}^2\xi\|_{2,-2}) \\ &\leq c\epsilon^{-2}\|\xi\|_{1,2,0} + \epsilon\|\mathring{\nabla}^2\xi\|_{2,-2}.\end{aligned} \tag{4.38}$$

By inserting these estimates back into (4.36), we obtain

$$\|\mathring{\nabla}^2\xi\|_{2,-5/2} \leq C\big(\|f_1\|_{2,-5/2} + \|(f_3,f_4)\|_{1,2,-3/2}\big) + c(\epsilon)\|\xi\|_{1,2,0} + \epsilon\|\mathring{\nabla}^2\xi\|_{2,-2}, \tag{4.39}$$

and then the weighted Poincaré and interpolation inequalities give

$$\|\xi\|_{2,2,-1/2} \leq C\big(\|f_1\|_{2,-5/2} + \|(f_3,f_4)\|_{1,2,-3/2}\big) + c(\epsilon)\|\xi\|_{2,0} + \epsilon\|\xi\|_{2,2,0}. \tag{4.40}$$

Now choosing ϵ sufficiently small completes the proof. □

Remark 4.9. While Lemma 4.8 gives an estimate on $\mathcal{M}$, the weighted Hölder, Sobolev and interpolation inequalities used above are also valid on A_R (see Remark 3.5). In particular, we have

$$\|\mathring{\nabla}^2\xi\|_{2,-5/2:A_R} \leq C\Big(\|f_1\|_{2,-5/2:A_R} + \|(f_3,f_4)\|_{1,2,-3/2:A_R} + \|\xi\|_{2,0:A_R}\Big) \tag{4.41}$$

for $\xi \in W^{2,2}_\delta(A_R)$, where C is independent of R. This is useful for the proof of Theorem 4.10, below.

Note that the bound for $\mathring{\nabla}^2 V$ is the first place where we require the $o(r^{-3/2})$ decay for $A_\mathfrak{t}$; in earlier estimates we could have relaxed this to $o(r^{-1})$. It should also be noted that $D\Phi^*_A$ makes no appearance in Lemma 4.8. In fact, the proof of Theorem 4.10 below requires no explicit restrictions on $D\Phi^*_A[\xi]$. However, the conditions on $D\Phi^*_g[\xi], D\Phi^*_\pi[\xi], D\Phi^*_\varepsilon[\xi]$ implicitly imply $D\Phi^*_A[\xi] \in L^2_{3/2}$. This is a consequence of Theorem 4.10 below, and Lemma 6.5 of Chapter 6.

Recall, for an arbitrary $\xi \in \mathcal{N}$, we can only interpret $D\Phi^*$ as a differential operator in the weak sense (see 4.27). Theorem 4.10, below, gives conditions under which we can interpret $D\Phi^*[\xi] = f$ as a differential equation in the usual sense. This is analogous to Proposition 3.5 in [11] and in fact, local regularity is established by an identical argument as we have an equation of the same form. We then make use of Lemma 4.8 to establish global regularity.

Theorem 4.10. *If $\xi \in \mathcal{N}$ is a weak solution of $D\Phi^*_{(g,A,\pi,\varepsilon)}[\xi] = (f_1, f_2, f_3, f_4)$, with $(f_1, f_3, f_4) \in L^2_{-5/2} \times W^{1,2}_{-3/2} \times W^{1,2}_{-3/2}$ and $(g, A, \pi, \varepsilon) \in \mathcal{F}$, then $\xi \in W^{2,2}_{-1/2}$ and is indeed a strong solution.*

Proof. $D\Phi^*$ is second order in N but only first order in both X and V, so we differentiate $D\Phi^*_\pi[\xi]$ and $D\Phi^*_\varepsilon[\xi]$ in order to consider N, X and V on equal footing. Explicitly, we define the operator, Ψ, by the following equation:

$$\Psi[\xi] := \begin{bmatrix} D\Phi^*_g[\xi] \\ \mathring{\nabla} D\Phi^*_\pi[\xi] \\ \mathring{\nabla} D\Phi^*_\varepsilon[\xi] \end{bmatrix} = \begin{bmatrix} f_1 \\ \mathring{\nabla} f_3 \\ \mathring{\nabla} f_4 \end{bmatrix}. \tag{4.42}$$

We first establish local regularity by restricting to a coordinate neighbourhood, Ω. In local coordinates, (4.42) is equivalent to an expression of the form

$$A \cdot \partial^2 \xi + B \cdot \partial \xi + \tilde{C} \cdot \xi = F,$$

where A is invertible. Explicit expressions of $\mathring{\nabla}^2 \xi$ can be seen in the proof of Lemma 4.8 and from this, it can be seen that the coefficients have regularity

$A \in W^{2,2}, B \in W^{1,2}, \tilde{C} \in L^2$. This is then equivalent to the expression,

$$\partial^2 \xi^\alpha + \partial_k (b^{k\alpha}_{ij\beta} \xi^\beta) + \tilde{c}^\alpha_{ij\beta} \xi^\beta = \tilde{f}^\alpha_{ij},$$

where $b \in W^{1,2}$ and $\tilde{c}, \tilde{f} \in L^2$. Therefore our weak solution, $\xi \in L^2$, must satisfy the weak form of the above expression;

$$\int_\Omega (\partial^2_{ij} \phi^{ij}_\alpha - b^{k\beta}_{ij\alpha} \partial_k \phi^{ij}_\beta + \tilde{c}^\beta_{ij\beta} \phi^{ij}_\alpha) \xi^\alpha dx = \int_\Omega \phi^{ij}_\alpha f^\alpha_{ij}, \tag{4.43}$$

for all compactly supported $\phi \in W^{2,2}(\Omega)$. For $\epsilon \ll 1$, let J_ϵ be a mollification operator, so that $J_\epsilon \phi \in C^\infty_c(\Omega)$. By inserting this mollified ϕ into (4.43) and noting $\partial(J_\epsilon \phi) = J_\epsilon(\partial \phi)$, we then have

$$\partial^2 \xi_\epsilon + \partial(J_\epsilon b \xi) + J_\epsilon(\tilde{c}\xi) = J_\epsilon \tilde{f} \qquad \text{on } \Omega, \tag{4.44}$$

in the strong sense, where $\xi_\epsilon := J_\epsilon \xi \in C^\infty$. If we take the trace of (4.44) then the highest order term is $\Delta_0 \xi_\epsilon$, the Euclidean Laplacian, and we may then estimate ξ_ϵ in terms of the fundamental solution of Laplace's equation. However, in order to avoid introducing a boundary integral on $\partial\Omega$, we introduce a smooth cut-off function, $\chi \in C^\infty_c(\Omega)$, and let $u = \chi \xi_\epsilon$. We then insert the trace of (4.44) into the expression,

$$\Delta_0 u = \chi \Delta_0 \xi_\epsilon + 2\partial\chi \cdot \partial \xi_\epsilon + \xi_\epsilon \Delta_0 \chi,$$

and arrive an an equation of the form

$$\Delta_0 u = F + \partial G,$$

where $F = F_1 + F_2 + F_3$ and $G = G_1 + G_2$, with $F_1 = \chi'' J_\epsilon \xi + \chi J_\epsilon \tilde{f}$, $F_2 = \chi' J_\epsilon(b\xi)$, $F_3 = \chi J_\epsilon(\tilde{c}\xi)$, $G_1 = \chi' J_\epsilon \xi$ and $G_2 = \chi J_\epsilon(b\xi)$. Since $F, G \in C^\infty_c(\Omega)$, we can explicitly write u as

$$u(x) = \Gamma * (F + \partial G) = \int_\Omega \Gamma(x-y)(F(y) + \partial G(y))dy, \tag{4.45}$$

where $\Gamma(x-y) = 1/(4\pi|x-y|)$ is the fundamental solution of Laplace's equation. Let $K_{ij} = \partial^2_{ij}\Gamma$, which is a Calderon-Zygmund kernel in the usual sense and therefore satisfies (see, for example, [56], Chapter II)

$$\|K_{ij} * w\|_p \leq c\|w\|_p, \tag{4.46}$$

where the norms are to be understood as integrals on Ω, here and in the other local estimates below.

We have $K_{ij} * F_1 = \partial^2_{ij}(\Gamma * F_1)$, and therefore the standard Poincaré inequality and (4.46) give

$$\|\Gamma * F_1\|_{2,2} \leq c\|F_1\|_2 \leq c\|\xi\|_2 + \|\tilde{f}\|_2, \tag{4.47}$$

since $J_\epsilon \xi \to \xi$ and $J_\epsilon \tilde{f} \to \tilde{f}$ in L^2. Similarly we have

$$\|\Gamma * F_2\|_{2,3/2} \leq c\|F_2\|_{3/2} \leq c\|b\|_6\|\xi\|_2 \leq c\|b\|_{1,2}\|\xi\|_2, \tag{4.48}$$

where we have again made use of the weighted Hölder and Sobolev inequalities. Note that we are not able to directly obtain a $W^{2,2}$ estimate for $\|\Gamma * F_2\|_{2,3/2}$, and the estimate for $\|\Gamma * F_3\|$ below is even weaker. However, we can bootstrap this up to the required $W^{2,2}$ estimate.

Let I_1 be the usual Riesz potential on Ω, defined by

$$I_1 u(x) = \int_\Omega \frac{u(y)}{|x-y|^2} dy. \tag{4.49}$$

The Riesz potential is the inverse of a first order pseudo-differential operator, and as such it satisfies the following Sobolev-type inequality (see, for example, Theorem 23.1 and Proposition 23.3, Chapter VIII of [28]):

$$\|I_1 u\|_p \leq c\|u\|_q, \tag{4.50}$$

for $1 < q < n = 3$ and $\frac{1}{p} = \frac{1}{q} - \frac{1}{3}$, or $q = 1$ and $1 < p < \frac{n}{n-1} = 3/2$. From this, and noting $\partial\Gamma * u \sim I_1 u$, we have

$$\|\partial\Gamma * F_3\|_p \leq c\|I_1 F_3\|_p \leq c\|F_3\|_1 \leq c\|\tilde{c}\|_2\|\xi\|_2,$$

for $1 < p < 3/2$. Then the standard Poincaré inequality gives us

$$\|\Gamma * F_3\|_s \leq c\|\partial\Gamma * F_3\|_p \leq c\|\tilde{c}\|_2\|\xi\|_2,$$

for $3/2 < s < 3$, where $p = \frac{3s}{3+s} < 3/2$. Now, by making use of the identity, $\Gamma * \partial G_1 = \partial(\Gamma * G_1)$, and the Sobolev and Poincaré inequalities, we have

$$\|\Gamma * \partial G_1\|_6 \leq c\|\partial(\Gamma * G_1)\|_{1,2} \leq c\|\partial^2\Gamma * G_1\|_2 \leq c\|G_1\|_2 \leq c\|\xi\|_2.$$

Similarly, we have

$$\begin{aligned}\|\Gamma * \partial G_2\|_3 &\leq c\|\partial(\Gamma * G_2)\|_{1,3/2} \leq c\|\partial^2\Gamma * G_2\|_{3/2} \\ &\leq c\|G_2\|_{3/2} \leq c\|b\|_6\|\xi\|_2 \leq c\|b\|_{1,2}\|\xi\|_2.\end{aligned}$$

Now that we have a uniform bound on u, independent of ϵ, we conclude $\xi \in L^{3-\epsilon}_{loc}$. This argument can be repeated successively, each time with a stronger estimate for ξ, until we eventually arrive at $\xi \in W^{2,2}_{loc}$. The remainder of the bootstrapping argument is included as Appendix B, as it is simply a matter of iterating the exact same estimates used above.

To establish global regularity, we introduce a new smooth cutoff function $\chi \in C^\infty_c(\mathcal{M})$ such that $\chi \equiv 1$ on B_{R_0}, for some $R_0 > 1$ and $\chi = 0$ on B_{2R_0}. Define $\chi_R(x) = \chi(xR_0/R)$, so that χ_R has support on B_{2R}. Clearly $\chi\xi_R \in W^{2,2}_{-1/2}$, therefore Lemma 4.8 gives

$$\begin{aligned}\|\chi_R\xi\|_{2,2,-1/2} \leq C\Big(&\|D\Phi^*_g[\chi_R\xi]\|_{2,-5/2} + \|D\Phi^*_\pi[\chi_R\xi]\|_{1,2,-3/2}\\ &+ \|D\Phi^*_\varepsilon[\chi_R\xi]\|_{1,2,-3/2} + \|\xi\|_{2,0}\Big),\end{aligned} \tag{4.51}$$

since $\chi_R\xi \to \xi$ in L^2_0, and from this we can show that $\chi_R\xi$ is uniformly bounded in $W^{2,2}_{-1/2}$.

Since we have $\xi \in W^{2,2}_{loc}$, we can interpret $D\Phi^*$ as a true differential operator acting on ξ. Also note that $\mathring{\nabla}\chi_R(x) = (R_0/R)\mathring{\nabla}\chi(xR_0/R)$, $\mathring{\nabla}\chi$ is bounded, and $\mathring{\nabla}\chi_R$ has support on A_R. It follows that we have

$$\|u\mathring{\nabla}\chi_R\|_{p,\delta} \leq c\|u/R\|_{p,\delta:A_R} \leq c\sup_{x\in A_R}|r(x)/R|\|u\|_{p,\delta+1:A_R} \leq c\|u\|_{p,\delta+1:A_R}.$$

From this and the usual weighted Sobolev-type inequalities, we have

$$\begin{aligned}\|D\Phi^*_g[\chi_R\xi]\|_{2,-5/2} &\leq c\Big(\|\chi_R D\Phi^*_g[\xi]\|_{2,2,-1/2} + \|\pi\xi\mathring{\nabla}\chi_R\|_{2,-5/2}\\ &\quad + \|\xi\mathring{\nabla}^2\chi_R\|_{2,-5/2} + \|\mathring{\nabla}(\xi)\mathring{\nabla}(\chi_R)\|_{2,-5/2}\Big)\\ &\leq c\Big(\|f_1\|_{2,-5/2} + \|\pi\|_{4,-3/2}\|\xi\|_{4,0:A_R} + \|\xi\|_{2,-1/2}\\ &\quad + \|\mathring{\nabla}\xi\|_{2,-3/2:A_R}\Big)\\ &\leq c\left(\|f_1\|_{2,-5/2} + \|\pi\|_{1,2,-3/2}\|\xi\|_{1,2,0:A_R} + \|\xi\|_{2,-1/2}\right.\\ &\quad \left.+ \|\mathring{\nabla}\xi\|_{2,-3/2:A_R}\right)\\ &\leq C\left(\|f_1\|_{2,-5/2} + \|\xi\|_{2,-1/2} + \|\mathring{\nabla}\xi\|_{2,-3/2:A_R}\right)\\ &\leq C\left(\|f_1\|_{2,-5/2} + \|\xi\|_{2,-1/2}\right) + \epsilon\|\mathring{\nabla}^2\xi\|_{2,-5/2:A_R}.\end{aligned}$$

Almost identically, we have

$$\begin{aligned}\|\mathring{\nabla} D\Phi^*_\pi[\chi_R\xi]\|_{2,-5/2} \leq \Big(&\|\chi_R\mathring{\nabla} D\Phi^*_\pi[\xi]\|_{2,-5/2} + \|\pi\xi\mathring{\nabla}\chi_R\|_{2,-5/2}\\ &+ \|\xi\mathring{\nabla}^2\chi_R\|_{2,-5/2} + \|\mathring{\nabla}(\xi)\mathring{\nabla}(\chi_R)\|_{2,-5/2}\Big)\\ \leq\, & C\left(\|\mathring{\nabla} f_3\|_{2,-5/2} + \|\xi\|_{2,-1/2}\right) + \epsilon\|\mathring{\nabla}^2\xi\|_{2,-5/2:A_R}.\end{aligned}$$

For the sake of presentation, we now gather terms with the same regularity and decay rate; we define $\sigma = (\varepsilon, \mathring{\nabla} A, A_{\mathfrak{k}}) \in W^{1,2}_{-3/2}$. Then we again have

$$\begin{aligned}\|\mathring{\nabla} D\Phi^*_\varepsilon[\chi_R\xi]\|_{2,-5/2} \leq \Big(&\|\chi_R\mathring{\nabla} D\Phi^*_\varepsilon[\xi]\|_{2,-5/2} + \|\sigma\xi\mathring{\nabla}\chi_R\|_{2,-5/2}\\ &+ \|\xi\mathring{\nabla}^2\chi_R\|_{2,-5/2} + \|\mathring{\nabla}(\xi)\mathring{\nabla}(\chi_R)\|_{2,-5/2}\\ &+ \|A\mathring{\nabla}(\xi)\mathring{\nabla}(\chi_R)\|_{2,-5/2}\Big)\\ \leq \Big(&\|\chi_R\mathring{\nabla} D\Phi^*_\varepsilon[\xi]\|_{2,-5/2} + \|\sigma\xi\mathring{\nabla}\chi_R\|_{2,-5/2}\\ &+ \|\xi\mathring{\nabla}^2\chi_R\|_{2,-5/2} + (1+\|A\|_{\infty,0})\|\mathring{\nabla}(\xi)\mathring{\nabla}(\chi_R)\|_{2,-5/2}\Big)\\ \leq\, & C\left(\|\mathring{\nabla} f_4\|_{2,-5/2} + \|\xi\|_{2,-1/2}\right) + \epsilon\|\mathring{\nabla}^2\xi\|_{2,-5/2:A_R}.\end{aligned}$$

Inserting the estimates above into (4.51) and applying the weighted Poincaré inequality, we arrive at

$$\begin{aligned}\|\mathring{\nabla}^2(\chi_R\xi)\|_{2,-5/2} \leq C\big(&\|f_1\|_{2,-5/2} + \|(f_3,f_4)\|_{1,2,-3/2}\\ &+ \|\xi\|_{2,-1/2} + \epsilon\|\mathring{\nabla}^2\xi\|_{2,-5/2:A_R}\big).\end{aligned} \tag{4.52}$$

Unfortunately we are unable to ensure $\|\mathring{\nabla}^2\xi\|_{2,-5/2:A_R} \lesssim \|\mathring{\nabla}^2(\chi_R\xi)\|_{2,-5/2}$, so we can not absorb the last term into the left-hand side of (4.52). However, recalling Remark 4.9, we apply the local version of Lemma 4.8 to obtain

$$\|\mathring{\nabla}^2\xi\|_{2,-5/2:A_R} \leq C\|f_1\|_{2,-5/2} + \|(f_3,f_4)\|_{1,2,-3/2} + \|\xi\|_{2,0}. \tag{4.53}$$

Finally we obtain the desired uniform bound:

$$\begin{aligned}\|\chi_R\xi\|_{2,2,-1/2} &\le c\|\mathring{\nabla}^2(\chi_R\xi)\|_{2,-5/2} \\ &\le C\big(\|f_1\|_{2,-5/2} + \|(f_3,f_4)\|_{1,2,-3/2} + \|\xi\|_{2,-1/2}\big).\end{aligned} \tag{4.54}$$

A well-known consequence of the Banach-Alaoglu theorem is that every bounded sequence in a Hilbert space has a weakly convergent subsequence; it follows that $\chi_R\xi$ converges to ξ weakly in $W^{2,2}_{-1/2}$. In particular, $\xi \in W^{2,2}_{-1/2}$ and satisfies $D\Phi^*[\xi] = (f_1, f_2, f_3, f_4)$ strongly. □

Recall that we wish to prove that $D\Phi$ is surjective, so we first establish that $D\Phi^*$ has trivial kernel. In light of Theorem 4.10, the equation $D\Phi^*[\xi] = 0$ can be interpreted as a bona fide differential equation. It then follows that the equation $\Psi[\xi] = 0$, where Ψ is defined by (4.42), is equivalent to an equation of the form

$$\mathring{\nabla}^2\xi = b_1\nabla\xi + b_0\xi, \tag{4.55}$$

where $b_0 \in L^2_{-5/2}$ and $b_1 \in W^{1,2}_{-3/2}$. This can easily be seen by examining the expressions for $\mathring{\nabla}^2 N$, $\mathring{\nabla}^2 X$ and $\mathring{\nabla}^2 V$ in the proof of Lemma 4.8. While the statement of Theorem 3.6 in [11] is only concerned with a particular equation, it is immediately obvious that the proof holds more generally. In particular, the theorem could be instead stated as follows:

Theorem 4.11 (Theorem 3.6 of [11])**.** *Let $\Omega \subset \mathcal{M}$ be a connected domain with $E'_R \subset \Omega$ for some R, where E'_R is a connected component of E_R. If $\xi \in W^{2,2}_{-1/2}$ satisfies*

$$\mathring{\nabla}^2\xi = b_1\nabla\xi + b_0\xi,$$

with $b_0 \in L^2_{-5/2}$ and $b_1 \in W^{1,2}_{-3/2}$, then $\xi \equiv 0$ in Ω.

From this and Theorem 4.10, we have the following immediate corollary.

Corollary 4.12. *Let* $(g, A, \pi, \varepsilon) \in \mathcal{F}$*. If* $\xi \in \mathcal{N}^*$ *satisfies* $D\Phi(g, A, \pi, \varepsilon)^*[\xi] = 0$ *on* $\mathcal{M}$ *then* $\xi \equiv 0$*.*

This establishes that $D\Phi^*_{(g,A,\pi,\varepsilon)}$ has a trivial kernel for all $(g, A, \pi, \varepsilon) \in \mathcal{F}$; that is, there are no symmetries of the data which are $o(r^{-1/2})$ at infinity. Note that if $D\Phi^*[\xi] = 0$ on $\mathcal{M}$ and ξ is $o(r^{-1/2})$ only on a single end, then Theorem 4.11 implies ξ vanishes on any connected region containing that end and therefore vanishes on all of $\mathcal{M}$.

4.5 The Constraint Submanifold and Linearisation Stability

In this Section, we prove the main result of this chapter: the level sets of Φ are Hilbert submanifolds of $\mathcal{F}$. We then briefly discuss this in relation to the property of linearisation stability.

In order to proceed, we restrict $D\Phi$ to particular variations, $(h, b, p, f) \in T_{(g,A,\pi,\varepsilon)}\mathcal{F}$, such that $D\Phi$ resembles an elliptic operator (cf. [26]). In particular, we write $(h, b, p, f) = \varphi(y, Y, \psi)$ as

$$\begin{aligned} h_{ij} &= -\frac{1}{2} g_{ij} y, & b_i^a &= 0, \\ p^{ij} &= \frac{1}{2}(\nabla^i Y^j + \nabla^i Y^j - \nabla_k Y^k g^{ij})\sqrt{g}, & f_i^a &= -\nabla_i \psi^a \sqrt{g}. \end{aligned}$$

Then $D\Phi[h,b,p,f] = D\Phi[\varphi(y,Y,\psi)] = F[y,Y,\psi]$ is given by

$$F[y,Y,\psi] =$$
$$\begin{bmatrix} \Delta y\sqrt{g} - \frac{1}{4}\Phi_0(g,A,\pi,\varepsilon)y + \frac{1}{2}\pi_j^j\nabla_j Y^j - 2\pi^{ij}\nabla_i Y_j - \frac{1}{4}(E^2+B^2)y + \varepsilon_a^i\partial_i\psi^a \\ \begin{pmatrix} \Delta Y_i\sqrt{g} + R_{ij}Y^j\sqrt{g} + \nabla^j(\psi_a)(\mathring{\nabla}_i A_j^a - \mathring{\nabla}_j A_i^a)\sqrt{g} - A_i^a\Delta\psi_a\sqrt{g} \\ \qquad -\nabla_j(\pi_i^j)y - \pi_i^j\mathring{\nabla}_j y + \frac{1}{2}\pi_j^j\mathring{\nabla}_i y \end{pmatrix} \\ \Delta\psi_a\sqrt{g} + C_{ab}^c\mathring{\nabla}^j(\psi_c)A_j^b\sqrt{g} \end{bmatrix}.$$

We require the following scale-broken estimate from [9], for operators that are asymptotic to the Laplacian.

Proposition 4.13. *For $\lambda > 0$, $n < q < \infty$ and $\tau \le 0$, let*

$$Pu = a^{ij}(x)\mathring{\nabla}_i\mathring{\nabla}_j u + b^i(x)\mathring{\nabla}_i u + \tilde{c}(x)u$$

satisfy

$$\lambda\mathring{g}^{ij}(x)v_iv_j < a^{ij}(x)v_iv_j < \lambda^{-1}\mathring{g}^{ij}(x)v_iv_j \qquad \text{for all } x\in\mathcal{M}, v\in T_x^*\mathcal{M},$$

$$\|a - \mathring{g}^{-1}\|_{1,q,\tau} + \|b\|_{q,\tau-1} + \|\tilde{c}\|_{q/2,\tau-2} \le C.$$

If $u \in L_\delta^p$ and $Pu \in L_{\delta-2}^2$, with $1 < p \le q$ and $\delta\in\mathbb{R}$, then $u \in W_\delta^{2,p}$ and satisfies

$$\|u\|_{2,p,\delta} \le C(\|Pu\|_{p,\delta-2} + \|u\|_{p:B_R}), \tag{4.56}$$

where R is fixed, independent of u, and B_R is as defined in Section 4.1

From this we establish a scale-broken estimate for F.

Lemma 4.14. *The map, $\varphi : W_{-1/2}^{2,2} \to T_{(g,A,\pi,\varepsilon)}\mathcal{F}$, and therefore also the map, $F : W_{-1/2}^{2,2} \to L_{-5/2}^2$, is a bounded a operator.*

Furthermore, for $\mathcal{Y} = (y, Y, \psi) \in W^{2,2}_{-1/2}$, *F satisfies the scale-broken estimate:*

$$\|\mathcal{Y}\|_{2,2,-1/2} \leq C(\|F[\mathcal{Y}]\|_{2,-5/2} + \|\mathcal{Y}\|_{2,0}). \tag{4.57}$$

Proof. The weighted Hölder and Sobolev inequalities give

$$\begin{aligned}
\|\mathring{\nabla}^2 h\|_{2,-5/2} &\leq c\left(\|g\|_{\infty,0}\|\mathring{\nabla}^2 y\|_{2,-5/2} + \|\mathring{\nabla}(g)\|_{4,-3/2}\|\mathring{\nabla} y\|_{4,-3/2}\right.\\
&\qquad \left.+ \|\mathring{\nabla}^2 g\|_{2,-5/2}\|y\|_{\infty,-1/2}\right)\\
&\leq C\,\|y\|_{2,2,-1/2},\\
\|\mathring{\nabla} p\|_{2,-5/2} &\leq C\left(\|\mathring{\nabla}^2 Y\|_{2-5/2} + \|\mathring{\nabla}(g)\mathring{\nabla} Y\|_{2,-5/2} + \|\tilde{\Gamma}\mathring{\nabla} Y\|_{2,-5/2}\right.\\
&\qquad \left.+ \|Y\mathring{\nabla}\tilde{\Gamma}\|_{2,-5/2} + \|Y\tilde{\Gamma}\mathring{\nabla} g\|_{2,-5/2}\right)\\
&\leq C\left(\|\mathring{\nabla}^2 Y\|_{2,-5/2} + (\|\mathring{\nabla} g\|_{4,-3/2} + \|\tilde{\Gamma}\|_{4,-3/2})\|\mathring{\nabla} Y\|_{4,-3/2}\right.\\
&\qquad \left.+ \|Y\|_{\infty,-1/2}(\|\mathring{\nabla}\tilde{\Gamma}\|_{2,-5/2} + \|\tilde{\Gamma}\|_{4,-3/2}\|\mathring{\nabla} g\|_{4.-3/2})\right)\\
&\leq C\,\|Y\|_{2,2,-1/2},\\
\|\mathring{\nabla}\psi\|_{2,-5/2} &\leq c\,(\|\mathring{\nabla}^2\psi\|_{2,-5/2} + \|\mathring{\nabla}\psi\|_{4,-3/2}\|\mathring{\nabla} g\|_{4,-3/2})\\
&\leq C\,\|\psi\|_{2,2,-1/2},
\end{aligned}$$

and then by the weighted Poincaré inequality, we have that φ is bounded. It follows immediately that F is also bounded.

Note now, that the Laplace-Beltrami operator clearly satisfies the hypotheses of Proposition 4.13, with $\tau = -\frac{1}{2}$ and $q = 4$. It follows that we have the scale-broken estimate for Δ,

$$\|u\|_{2,2,-1/2} \leq C(\|\Delta u\|_{2,-5/2} + \|u\|_{2,0}). \tag{4.58}$$

The scale-broken estimate for F is obtained by comparing F to the Laplacian. We write $F[\mathcal{Y}] = (F_1, F_2, F_3)$, for the sake of presentation, and bound the terms separately:

$$\begin{aligned}
\|\Delta y\|_{2,-5/2} \leq & c\big(\|F_1\|_{2,-5/2} + \|y\|_{\infty,0}(\|\Phi_0\|_{2,-5/2} + \|\varepsilon^2\|_{2,-5/2} + \|B^2\|_{2,-5/2}) \\
& + \|\nabla Y\|_{3,-1}\|\pi\|_{6,-3/2} + \|\mathring{\nabla}\psi\|_{3,-1}\|\varepsilon\|_{6,-3/2}\big) \\
\leq & c\big(\|F_1\|_{2,-5/2} + \|y\|_{\infty,0}(\|\Phi_0\|_{2,-5/2} + \|\varepsilon^2\|_{2,-5/2} + \|B^2\|_{2,-5/2}) \\
& + \|\mathring{\nabla} Y\|_{3,-1}\|\pi\|_{1,2,-3/2} + \|\tilde{\Gamma} Y\|_{3,-1}\|\pi\|_{1,2,-3/2} + \|\mathring{\nabla}\psi\|_{3,-1}\|\varepsilon\|_{1,2,-3/2}\big) \\
\leq & c\big(\|F_1\|_{2,-5/2} + \|y\|_{\infty,0}(\|\Phi_0\|_{2,-5/2} + \|\varepsilon^2\|_{2,-5/2} + \|B^2\|_{2,-5/2}) \\
& + \|\mathring{\nabla} Y\|_{3,-1}\|\pi\|_{1,2,-3/2} + \|\mathring{\nabla} g\|_{1,2,-1}\|Y\|_{\infty,0}\|\pi\|_{1,2,-3/2} \\
& + \|\mathring{\nabla}\psi\|_{3,-1}\|\varepsilon\|_{1,2,-3/2}\big).
\end{aligned}$$

The norms of $\Phi_0, \varepsilon, B, \pi$ and $\mathring{\nabla} g$, appearing above, are all finite and can be merged into the constant, C. Then making use of (4.37) and (4.38), we have

$$\|\Delta y\|_{2,-5/2} \leq C\|F_1\|_{2,-5/2} + C(\epsilon)\|\mathcal{Y}\|_{1,2,0} + \epsilon\|\mathring{\nabla}^2\mathcal{Y}\|_{2,-2}, \tag{4.59}$$

where $C(\epsilon)$ also depends on the point, $(g, A, \pi, \varepsilon) \in \mathcal{F}$. We bound $\Delta\psi$ and ΔY similarly as follows:

$$\begin{aligned}
\|\Delta\psi\|_{2,-5/2} \leq & c\,(\|F_3\|_{2,-5/2} + \|\nabla\psi A_{\mathfrak{k}}\|_{2,-5/2}) \\
\leq & c\,(\|F_3\|_{2,-5/2} + \|\mathring{\nabla}\psi\|_{3,-1}\|A_{\mathfrak{k}}\|_{6,-3/2} + \|\tilde{\Gamma}\|_{3,-1}\|A_{\mathfrak{k}}\|_{6,-3/2}\|\psi\|_{\infty,0}) \\
\leq & c\,(\|F_3\|_{2,-5/2} + \|\mathring{\nabla}\psi\|_{3,-1}\|A_{\mathfrak{k}}\|_{1,2,-3/2} \\
& + \|\mathring{\nabla} g\|_{1,2,-3/2}\|A_{\mathfrak{k}}\|_{1,2,-3/2}\|\psi\|_{\infty,0}).
\end{aligned}$$

From this we have the following estimate for $\Delta\psi$, almost identical to (4.59):

$$\|\Delta\psi\|_{2,-5/2} \leq c\|F_3\|_{2,-5/2} + C(\epsilon)\|\psi\|_{1,2,0} + \epsilon\|\mathring{\nabla}^2\psi\|_{2,-2}. \tag{4.60}$$

We make use of (4.60) in the following estimate for ΔY, as F_2 contains both ΔY and $\Delta\psi$ terms:

$$\begin{aligned}\|\Delta Y\|_{2,-5/2} \leq & c\Big(\|F_2\|_{2,-5/2} + \|\mathcal{Y}\|_{\infty,0}(\|Ric\|_{2,-5/2} + \|\nabla\pi\|_{2,-5/2}) \\ & + \|\mathring{\nabla}\mathcal{Y}\|_{3,-1}(\|\mathring{\nabla}A\|_{6,-3/2} + \|\pi\|_{6,-3/2}) + \|A\|_{\infty,0}\|\Delta\psi\|_{2,-5/2}\Big) \\ \leq & c\|F_2\|_{2,-5/2} + C(\|\mathcal{Y}\|_{\infty,0} + \|\mathring{\nabla}\mathcal{Y}\|_{3,-1} + \|\Delta\psi\|_{2,-5/2}).\end{aligned} \tag{4.61}$$

Recall when we defined Φ, we noted that the momentum constraint sometimes is defined differently. Had we used the alternative definition of Φ_i, then F_2 wouldn't contain the term $\Delta\psi$, and we would have $F(\mathcal{Y})$ is exactly $\Delta\mathcal{Y}$ plus lower order terms [46]. However, our definition of the momentum constraint is better suited to the results in Chapter 6.

Now, combining the estimates, (4.59)-(4.61), we conclude

$$\|\Delta\mathcal{Y}\|_{2,-5/2} \leq C\|F[\mathcal{Y}]\|_{2,-5/2} + C(\epsilon)\|\mathcal{Y}\|_{1,2,0} + \epsilon\|\mathring{\nabla}^2\mathcal{Y}\|_{2,-2}, \tag{4.62}$$

for any $\epsilon > 0$. By inserting this into the scale-broken estimate for Δ (4.58), we have

$$\|\mathcal{Y}\|_{2,2,-1/2} \leq C\|F[\mathcal{Y}]\|_{2,-5/2} + C(\epsilon)\|\mathcal{Y}\|_{1,2,0} + \epsilon\|\mathring{\nabla}^2\mathcal{Y}\|_{2,-2}. \tag{4.63}$$

The weighted interpolation and Poincaré inequalities then give

$$\|\mathcal{Y}\|_{2,2,-1/2} \leq C\|F[\mathcal{Y}]\|_{2,-5/2} + C(\epsilon)\|\mathcal{Y}\|_{2,0} + \epsilon\|\mathring{\nabla}^2\mathcal{Y}\|_{2,-2}. \tag{4.64}$$

Finally, choosing ϵ sufficiently small, we arrive at the scale-broken estimate for F:

$$\|\mathcal{Y}\|_{2,2,-1/2} \leq C(\|F[\mathcal{Y}]\|_{2,-5/2} + \|\mathcal{Y}\|_{2,0}). \tag{4.65}$$

□

We are now in a position to prove the main Theorem of this Chapter.

Theorem 4.15. *For all $s \in \mathcal{N}^*$, the level set*

$$\mathcal{C}(s) := \{(g, A, \pi, \varepsilon) \in \mathcal{F} \,:\, \Phi(g, A, \pi, \varepsilon) = s\} \tag{4.66}$$

is a Hilbert submanifold of $\mathcal{F}$.

Proof. As mentioned in Section 4.4, this is established by demonstrating that the hypotheses of Theorem 4.7 are satisfied. Since $D\Phi$ is bounded and linear, the kernel is closed and therefore is trivially complementable, due to the Hilbert structure. We only must prove that $D\Phi$ is surjective onto $\mathcal{N}^*$, and the conclusion follows.

Since the codomain splits as $\overline{\mathrm{ran}(D\Phi)} \oplus \mathrm{coker}(D\Phi)$ and $\mathrm{coker}(D\Phi) \simeq \ker(D\Phi^*)$ is trivial, all that remains to be shown is that $D\Phi$ has closed range; this is where we make use of the operator F. Clearly $\mathrm{ran}(F) \subset \mathrm{ran}(D\Phi)$, so it will suffice to prove that $\mathrm{ran}(F)$ is closed and differs from $\mathrm{ran}(D\Phi)$ by a finite dimensional closed subspace.

Lemma 4.14 allows us to employ the following standard argument, proving that F has finite dimensional kernel. Take any sequence $\mathcal{Y}_n \in \ker(F)$ such that $\|\mathcal{Y}_n\|_{2,2,-1/2} \leq 1$; a sequence in the closed unit ball in $\ker F$. By the weighted version of the Rellich compactness theorem (Proposition 3.7), the closed $W^{2,2}_{-1/2}$-unit ball is compact with respect to the L^2_0 topology. It then follows that $\mathcal{Y}_n$ has a subsequence that converges in L^2_0, but since equation (4.57) implies that $\|\mathcal{Y}_n - \mathcal{Y}_m\|_{2,2,-1/2} \leq C\|\mathcal{Y}_n - \mathcal{Y}_m\|_{2,0}$, the subsequence also converges in $W^{2,2}_{-1/2}$. By the sequential characterisation of compactness, it follows that the closed unit ball in $\ker(F)$ is compact and thus $\ker(F)$ is finite dimensional.

Since the kernel is closed, we have the decomposition, $W^{2,2}_{-1/2} = K \oplus \ker(F)$, where K is some closed subspace. To prove that the range of F is closed, we require the following estimate:

$$\|\mathcal{Y}\|_{2,2,-1/2} \leq c\|F[\mathcal{Y}]\|_{2,-5/2} \qquad \text{for all } \mathcal{Y} \in K. \tag{4.67}$$

We prove this by contradiction. If (4.67) were not true then we could find a sequence $\mathcal{Y}_n \in K$ such that $\|\mathcal{Y}_n\|_{2,2,-1/2} = 1$ and $\|F[\mathcal{Y}_n]\|_{2,-5/2} \to 0$. Then, passing to a subsequence again if necessary, the weighted Rellich compactness theorem implies $\mathcal{Y}_n \to \mathcal{Y} \neq 0$ with respect to the L^2_0 topology. Then from (4.57) we also have convergence with respect to the $W^{2,2}_{-1/2}$ norm. Since K is closed in $W^{2,2}_{-1/2}$ we conclude $0 \neq \mathcal{Y} \in K \cap \ker(F)$, which is a contradiction.

Now take any Cauchy sequence $F[\mathcal{Y}_n] \in \operatorname{ran}(F)$, which necessarily converges to some $Q \in L^2_{-1/2}$. Then (4.67) implies $\mathcal{Y}_n$ converges to some $\mathcal{Y} \in W^{2,2}_{-1/2}$. Since F is a bounded linear operator, it follows that $F[\mathcal{Y}_n] \to F[\mathcal{Y}]$ and therefore $Q = F[\mathcal{Y}] \in \operatorname{ran}(F)$; that is, F has closed range.

It remains to be shown that $\operatorname{ran}(F)$ differs from $\operatorname{ran}(D\Phi)$ by a finite dimensional subspace. Given that we have the splitting

$$L^2_{-1/2} = \overline{\operatorname{ran}(D\Phi)} = \operatorname{ran}(F) \oplus \operatorname{coker}(F),$$

we show that $\ker(F^*)$ is finite dimensional.

The formal L^2 adjoint is given by

$$F^*[z, Z, \zeta] =$$
$$\begin{bmatrix} \Delta z\sqrt{g} - \frac{1}{4}\Phi_0(g, A, \pi, \varepsilon)z - \frac{1}{4}(E^2 + B^2)z + \pi_i^j \mathring{\nabla}_j Z^i - \frac{1}{2}\mathring{\nabla}_i(\pi_j^j Z^i) \\ \Delta Z_j\sqrt{g} + 2\nabla_i(\pi_j^i z) - \frac{1}{2}\nabla_j(\pi_i^i z) + R_{ij}Z^i\sqrt{g} \\ \begin{pmatrix} \Delta\zeta^a\sqrt{g} - \mathring{\nabla}_i(z\varepsilon_a^i) - \Delta(Z^i)A_i^a\sqrt{g} - \nabla^j(Z^i)(\nabla_i A_j^a + \nabla_j A_i^a)\sqrt{g} \\ \qquad + C_{ab}^c \mathring{\nabla}^j(A_j^b \zeta_c \sqrt{g}) - Z^i \nabla^j \nabla_i A_j^a \sqrt{g} \end{pmatrix} \end{bmatrix}.$$

We define a new operator, $\hat{F}^*$, by adding $A^a_j F^{*j}_2$ to F^*_3, where F_2 and F_3 refer to the second and third components of the operator F, respectively.

$$\hat{F}^*[z, Z, \zeta] =$$
$$\begin{bmatrix} \Delta z\sqrt{g} - \frac{1}{4}\Phi_0(g, A, \pi, \varepsilon)z - \frac{1}{4}(E^2 + B^2)z + \pi^j_i \mathring{\nabla}_j Z^i - \frac{1}{2}\mathring{\nabla}_i(\pi^j_j Z^i) \\ \Delta Z_j\sqrt{g} + 2\nabla_i(\pi^i_j z) - \frac{1}{2}\nabla_j(\pi^i_i z) + R_{ij} Z^i \sqrt{g} \\ \begin{pmatrix} \Delta\zeta^a\sqrt{g} - \mathring{\nabla}_i(z\varepsilon^i_a) + (2\nabla_i(\pi^i_j z) - \frac{1}{2}\nabla_j(\pi^i_i z) + R_{ij}Z^i\sqrt{g})A^{aj} \\ -\nabla^j(Z^i)(\nabla_i A^a_j + \nabla_j A^a_i)\sqrt{g} + C^c_{ab}\mathring{\nabla}^j(A^b_j\zeta_c\sqrt{g}) - Z^i\nabla^j\nabla_i A^a_j\sqrt{g} \end{pmatrix} \end{bmatrix}$$

This new operator is elliptic and clearly has the same kernel as F^*. Furthermore, it can be seen that $\hat{F}^*$ satisfies the hypotheses of Proposition 4.13 directly, again with $\tau = -\frac{1}{2}$ and $q = 4$. In particular, if $\hat{F}^*[z, Z, \zeta] \in L^2_{-5/2}$ then we have $(z, Z, \zeta) \in W^{2,2}_{-1/2}$ and $\hat{F}^*$ also satisfies the scale-broken estimate,

$$\|(z, Z, \zeta)\|_{2,2,-1/2} \leq C(\|F[z, Z, \zeta]\|_{2,-5/2} + \|(z, Z, \zeta)\|_{2,0}). \tag{4.68}$$

From this, we infer that $\ker(F^*) = \ker(\hat{F}^*)$ is finite dimensional by the exact same argument used to prove F has finite dimensional kernel. For the reasons outlined above, we conclude that $D\Phi$ has closed range and therefore the hypotheses of Theorem 4.7 are satisfied, which completes the proof. □

Theorem 4.15 not only provides us with the Hilbert manifold structure required for the following chapters, but it is also an interesting result in itself. We have that the Einstein-Yang-Mills, and therefore also the Einstein-Maxwell, constraint equations are linearisation stable in the sense of Section 2.4. In the earlier work of Arms [1, 2], Fischer and Marsden [32], linearisation stability of the full equations is inferred from the linearisation stability of the constraints.

This conclusion requires that the Cauchy problem is well-posed, which as mentioned at the end of Section 2.2, is expected to be true in the case considered here. Then the exact same argument used by Fischer, Marsden and Arms applies, to obtain linearisation stability of the full equations (see [2, 33]). However, as the existence and uniqueness results of Klainerman, Rodnianski and Szeftel [44] are rather technical, and linearisation stability is somewhat tangential to the main point of this thesis, we will not discuss this any further.

Chapter 5

Mass, Charge and Angular Momentum

The Higgs boson walks into a church. The priest says "we don't allow Higgs bosons in here".
The Higgs boson says "but without me, how can you have mass?"

Brian Malow (Science Comedian)

This chapter introduces the usual definitions of energy, momentum and charge as well as a generalised angular momentum quantity, which is suited to our purposes. These quantities are then expressed as volume integrals of divergences over the initial data manifold, rather than as surface integrals at infinity. We prove that these quantities are smooth maps acting on the constraint submanifold; a requirement for the Lagrange multiplier arguments of Chapter 6.

The ADM energy-momentum [5] is the standard definition of energy-momentum at spatial infinity for asymptotically flat spacetimes. The ADM energy is usually given by the expression,

$$E = \oint_{\infty} \partial_j g_{ij} - \partial_i g_{jj} dS_i, \tag{5.1}$$

where several things are implicitly understood: the notation $\oint_\infty$ refers to the limit, $\lim_{R\to\infty}\oint_{S_R}$, of integrals over large coordinate spheres of radius R, repeated indices are summed over, and the integral is evaluated using a coordinate system that is Cartesian at infinity. The coordinate conditions can easily be removed by writing the expression in terms of the background Euclidean metric as

$$E := \frac{1}{16\pi}\oint_\infty \mathring{\nabla}^j g_{ij} - \mathring{\nabla}_i(\mathring{g}^{jk}g_{jk})dS^i. \tag{5.2}$$

The linear momentum is a covector at spatial infinity, given by

$$p_i := \frac{1}{8\pi}\oint_\infty \pi_{ij}dS^j, \tag{5.3}$$

and together these quantities define the energy-momentum covector, (E, p_i), on the spacetime. This definition of momentum motivates the nomenclature, "momentum constraint", for (4.14). It has been established by Bartnik that these definitions are independent of the limiting process [9, 11].

While these definitions do depend on the choice of initial data slice, this is not in any way problematic, and is in fact to be expected. The energy-momentum covector is "at infinity", so may be identified with a 4-vector in Minkowski spacetime. Choosing different initial data slices is akin to choosing different coordinates for Minkowski spacetime; we can exchange energy for momentum, but the length of the 4-covector is invariant. Assuming the dominant energy condition, the positive mass theorem ensures that $(E, p_i)^\sharp$ is timelike, where $\sharp$ denotes the usual musical isomorphism identifying a covector with a vector. We therefore define the total mass as $m := \sqrt{E^2 - |p|^2}$.

Angular momentum in general relativity is unfortunately somewhat more complicated than its linear counterpart. Under the assumption of axially symmetry, we have the advantage of having a rotational Killing vector field, ϕ, whose orbits are 2π-periodic. In this case, the angular momentum is usually given by

$$J = -\frac{1}{8\pi}\oint_{\infty} \phi^i \pi_{ij} dS^j, \tag{5.4}$$

which is now a well-established definition. However, in the general case there is no rotational Killing vector field so this definition does not make sense. There is a notion of angular momentum, which does make sense more generally, however it also has limitations. Since there is no preferred direction of rotation, rotational vector fields are introduced using Cartesian coordinates near infinity and angular momentum corresponding to "rotations about each coordinate axis" is defined. In terms of these Cartesian coordinates, define

$$\phi_x = y\partial_z - z\partial_y, \qquad \phi_y = z\partial_x - x\partial_z, \qquad \phi_z = x\partial_y - y\partial_x, \tag{5.5}$$

which are the rotational symmetries of flat space. Then we define the three angular momenta associated with these symmetries,

$$J_{(x_i)} = -\frac{1}{8\pi}\oint_{\infty} \phi^j_{(x_i)} \pi_{jk} dS^k, \tag{5.6}$$

where $x_i = (x, y, z)$. The main problems with this definition are as follows (cf. [42]):

1. The integrals (5.6) do not converge in general, so often somewhat unnatural parity conditions are imposed [50] to ensure convergence.

2. The quantity $J_i = (J_{(x_1)}, J_{(x_2)}, J_{(x_3)})$ depends on the coordinate choice in a non-covariant fashion.

We avoid the problem of convergence by considering a quasilocal notion of angular momentum. Assume $\mathcal{M}$ has multiple asymptotic ends and fix a particular end, M_0, to work on. We then introduce a closed 2-surface boundary, Σ, such that $\mathcal{M} \setminus \Sigma$ comprises of two disconnected components, one of which contains M_0. The surface, Σ, will later play the role of the horizon in the first law. The component of $\mathcal{M}$ containing M_0 will be denoted by $\mathcal{M}_0$.

Note that in the vacuum axially symmetric case, we have $\nabla^i \pi_{ij} = 0$, and therefore $\nabla^i(\phi^j \pi_{ij}) = \pi_{ij}\nabla^{(i}\phi^{j)} + \phi^j \nabla^i \pi_{ij} = 0$ since ϕ is Killing. In particular, the divergence theorem gives

$$J = -\frac{1}{8\pi}\oint_\infty \phi^i \pi_{ij} dS^j = -\frac{1}{8\pi}\oint_\Sigma \phi^i \pi_{ij} dS^j. \tag{5.7}$$

Throughout, dS^i will refer to the surface element whose associated normal points in the direction of infinity in M_0.

Equation (5.7) motivates our quasilocal generalised definition of angular momentum[1]. For some vector field χ, defined locally near Σ, we define the generalised χ-angular momentum:

$$\tilde{J}^0_\chi = -\frac{1}{8\pi}\oint_\Sigma \chi^i \pi_i^j dS_j. \tag{5.8}$$

Note that this definition agrees with the usual definition when χ is the axial Killing vector, however $\tilde{J}^0_\chi$ is well-defined on the entire phase space.

Since we are dealing with the Einstein-Yang-Mills equations, we also would like to include a term corresponding to the angular momentum of the Yang-Mills fields. In this case the χ-angular momentum becomes

$$\tilde{J}_\chi = -\frac{1}{16\pi}\oint_\Sigma \left(2\chi^i \pi_i^j - \varepsilon_a^j A_i^a \chi^i\right) dS_j. \tag{5.9}$$

The term corresponding to the angular momentum of the Yang-Mills fields appears to have first been considered in [57]. Clearly this definition is not useful for describing angular momentum when χ is permitted to be an arbitrary vector field, however we will later impose conditions on χ in order to ensure that this quantity better describes some notion of angular momentum.

[1]While this is useful for our purposes, we do not argue here that this gives a suitable quasilocal definition of angular momentum in general. There is a great deal of literature on the problem of quasilocal mass and angular momentum (see [58] and references therein).

We next introduce the definition of Yang-Mills charge. The total charge is usually given by

$$Q_{\infty\, a} = \oint_{\infty} *F_a, \tag{5.10}$$

where $*F$ is the spacetime Hodge dual of the field strength 2-form. It is more conveniently expressed in terms of the initial data as

$$Q_{\infty\, a} = \frac{1}{4\pi} \oint_{\infty} E^i_a dS_i. \tag{5.11}$$

In the Maxwell electrovac case this integral can be pushed onto Σ, as was done with the angular momentum, since $\nabla_i E^i = 0$. However in the source-free Yang-Mills case, $\nabla_i E^i_a$ does not vanish and in fact is equal to $[E, A]_a$, so the integral over Σ differs from that at infinity. This may be interpreted as the total charge of a black hole spacetime differing from the charge of the black hole itself. That is, the charge is not necessarily from a singularity and in fact, the field itself may be charged[2]. This leads us to consider a quasilocal charge defined analogously:

$$Q_{\Sigma\, a} = \frac{1}{4\pi} \oint_{\Sigma} E^i_a dS_i. \tag{5.12}$$

It is straightforward to show that these quasilocal quantities are smooth maps on $\mathcal{F}$.

Proposition 5.1. *For $\chi \in L^\infty(\Sigma)$, the maps $Q_\Sigma : \mathcal{F} \to \mathfrak{g}^*$ and $\tilde{J}^0_\chi$, $\tilde{J}_\chi : \mathcal{F} \to \mathbb{R}$ are smooth.*

Proof. By considering any function $\varphi \in C^\infty_c(\mathcal{M})$ with $\varphi \equiv 1$ on Σ, the Sobolev trace theorem gives

$$|Q_\Sigma| \le c\|E\|_{L^1(\Sigma)} = \|\varphi E\|_{L^1(\Sigma)} \le c\|\varphi E\|_{L^2(\Sigma)} \le c\|E\|_{1,2,-3/2}. \tag{5.13}$$

[2]The fact that the field itself may be charged is related to the terminology "self-interacting", used by physicists to describe non-commutative gauge theories.

We can estimate $\tilde{J}^0_\chi$ and $\tilde{J}_\chi$ similarly:

$$\tilde{J}^0_\chi \leq c\|\chi\|_{L^2(\Sigma)}\|\varphi\pi\|_{L^2(\Sigma)} \leq c\|\chi\|_{L^2(\Sigma)}\|\pi\|_{1,2,-3/2},$$

$$\begin{aligned}\tilde{J}_\chi &\leq c(\|\chi\|_{L^2(\Sigma)}\|\pi\|_{1,2,-3/2} + \|\chi\|_{L^\infty(\Sigma)}\|\varphi A\|_{L^2(\Sigma)}\|\varphi\varepsilon\|_{L^2(\Sigma)}) \\ &\leq c\|\chi\|_{L^\infty(\Sigma)}(\|\pi\|_{1,2,-3/2} + \|A\|_{1,2,-1/2}\|\varepsilon\|_{1,2,-3/2}).\end{aligned}$$

Since Q_Σ, $\tilde{J}^0_\chi$, $\tilde{J}_\chi$ are bounded and linear, smoothness follows. □

It will be convenient to rewrite the surface integrals at infinity as integrals of total divergences over the whole manifold. For this, we will consider two different cases for the initial data manifold:

(a) $\mathcal{M}$ is an asymptotically flat manifold with a one asymptotic end and no interior boundaries.

(b) $\mathcal{M}_0$ is an asymptotically flat manifold with one asymptotic end and a closed interior boundary surface, Σ.

The restriction to a single end in case (a) is primarily for the sake of presentation. The reader may note that permitting multiple ends makes only the most superficial difference to the analysis and will only be significant in the interpretation of Theorem 6.6 and Corollary 6.7. We briefly discuss the case where many asymptotic ends are present in Section 6.4.

In case (a), the integral of a divergence will only give the surface integrals at infinity; however, in case (b), we obtain integrals on Σ corresponding to the quasilocal quantities. In both cases, the energy, momentum and charge at infinity are smooth functions on the constraint submanifolds, however the proof is slightly different in each case.

Let $\xi^\mu_\infty \in \mathbb{R}^{3+1}$ be identified with some timelike vector, corresponding to the tangent of the worldline of an observer at infinity, and let $\xi^a_\infty \in \mathfrak{g}$ correspond to the asymptotic value of the electric potential. A total measure of the energy viewed by this observer is then given by $\xi_\infty \cdot (E, p_i, Q_a)$, which will be more convenient to work with than (E, p_i, Q_a). In order to write this as the integral of a divergence, we need to make sense of extending ξ_∞ to a section of P, the principal G-bundle.

Near infinity, $\xi_\infty \in \mathbb{R}^{3,1} \oplus \mathfrak{g}$ may be identified with some smooth section, $\tilde{\xi}_\infty \in C^\infty(\Lambda^0(\mathcal{M}) \times T\mathcal{M} \times \mathfrak{g} \otimes \Lambda^0(\mathcal{M}))$, such that $\mathring{\nabla}\tilde{\xi}_\infty = 0$. We then say a smooth section, $\hat{\xi}_\infty \in C^\infty(\Lambda^0(\mathcal{M}) \times T\mathcal{M} \times \mathfrak{g} \otimes \Lambda^0(\mathcal{M}))$, is a *constant translation near infinity* representing ξ_∞ if $\hat{\xi}_\infty = \tilde{\xi}_\infty$ on E_{2R} and vanishes on B_R, for some R. While a representation of ξ_∞ is not unique, two distinct representations differ only by an element of $C^\infty_c(\Lambda^0(\mathcal{M}) \times T\mathcal{M} \times \mathfrak{g} \otimes \Lambda^0(\mathcal{M})) \subset W^{k,p}_\delta$. In particular, the spaces

$$L^p_{\xi_\infty} := \{\xi : \xi - \hat{\xi}_\infty \in L^p_{-1/2}(\Lambda^0(\mathcal{M}) \times T\mathcal{M} \times \mathfrak{g} \otimes \Lambda^0(\mathcal{M}))\}, \tag{5.14}$$

$$W^{k,p}_{\xi_\infty} := \{\xi : \xi - \hat{\xi}_\infty \in W^{k,p}_{-1/2}(\Lambda^0(\mathcal{M}) \times T\mathcal{M} \times \mathfrak{g} \otimes \Lambda^0(\mathcal{M}))\}, \tag{5.15}$$

of asymptotic translations are well-defined.

We now define a quantity $\mathbb{P}$, in terms of its pairing with ξ_∞, which will allow us to write (E, p_i, Q_a) as integrals of divergences over $\mathcal{M}$:

$$\begin{aligned} 16\pi\xi^0_\infty \mathbb{P}_0(g) = \int_{\mathcal{M}} \Big(& \xi^0_{\text{ref}} (\mathring{g}^{ki}\mathring{g}^{jl}\mathring{\nabla}_k\mathring{\nabla}_l g_{ij} - \mathring{\Delta}\text{tr}_{\mathring{g}} g) \\ & + \mathring{g}^{ki}\mathring{g}^{jl}\mathring{\nabla}_k\xi^0_{\text{ref}}(\mathring{\nabla}_l g_{ij} - \mathring{\nabla}_i \mathring{g}_{jl})\Big)\sqrt{\mathring{g}}, \end{aligned} \tag{5.16}$$

$$16\pi\xi^i_\infty \mathbb{P}_i(\pi) = 2\int_{\mathcal{M}} \left(\hat{\xi}^i_\infty \mathring{\nabla}_j \pi^j_i + \pi^j_i \mathring{\nabla}_j \hat{\xi}^i_\infty\right), \tag{5.17}$$

$$16\pi\xi^a_\infty \mathbb{P}_a(\varepsilon) = 4\int_{\mathcal{M}} \left(\hat{\xi}^a_\infty \mathring{\nabla}_i E^i_a + E^i_a \mathring{\nabla}_i \hat{\xi}^a_\infty\right). \tag{5.18}$$

Note that these definitions are independent of the choice of representation $\hat{\xi}_\infty$, as a change of $\hat{\xi}_\infty$ corresponds to changing the integrands by a total divergence with compact support. We may identify $\mathbb{P}_\alpha = (\mathbb{P}_0, \mathbb{P}_i, \mathbb{P}_a)$ with the asymptotic value of a section of TP.

The smoothness of $\mathbb{P}_0$ and $\mathbb{P}_i$ has previously been established by Bartnik [11]; however, for completeness we include a proof of this with the proof that $\mathbb{P}_a$ is smooth. Part of the proof involves establishing the following estimate, which is included separately as it will also be useful for the proof of Theorem 6.3 in the next chapter.

Lemma 5.2. *Suppose $g \in \mathcal{G}^+_\lambda$ for some $\lambda > 0$, then*

$$\|R - (\mathring{\nabla}^i\mathring{\nabla}^j g_{ij} - \mathring{\Delta} tr_{\mathring{g}} g)\|_{1,-3} \leq c(\lambda)(1 + \|g - \mathring{g}\|^2_{2,2,-1/2}). \tag{5.19}$$

Proof. Simply note that the difference $R - (\mathring{\nabla}^i\mathring{\nabla}^j g_{ij} - \mathring{\Delta}\mathrm{tr}_{\mathring{g}} g)$ has the form

$$\mathring{R} + (g - \mathring{g})^{-1}(Ric(g) + g^{-1}\mathring{\nabla}^2 g) + (g^{-1})^3(\mathring{\nabla} g)^2;$$

the full expression can be seen in Appendix A. We then have

$$\begin{aligned}\|R - (\mathring{\nabla}^i\mathring{\nabla}^j g_{ij} - \mathring{\Delta}\mathrm{tr}_{\mathring{g}} g)\|_{1,-3} \leq c(\lambda) \big(&\|\mathring{R}\|_{1,-3} + \|g - \mathring{g}\|_{2,-1/2}\|Ric\|_{2,-5/2} \\ &+ \|g - \mathring{g}\|_{2,-1/2}\|\mathring{\nabla}^2 g\|_{2,-5/2} + \|(\mathring{\nabla} g)^2\|_{2,-3}\big) \\ \leq c(\lambda) \big(&1 + \|g - \mathring{g}\|^2_{2,2,-1/2}\big),\end{aligned}$$

where we have made use of Proposition 4.2. □

Theorem 5.3. *If $s \in L^1$ and $\xi \in L^\infty$, then $\mathbb{P} : \mathcal{C}(s) \to \mathbb{R}^{3,1} \oplus \mathfrak{g}$ is a smooth map.*

Proof. Note that $L^1 = L^1_{-3}$ and $L^\infty = L^\infty_0$. Since $\mathbb{P}$ is linear in (g, A, π, ε), it suffices to prove that it is locally bounded, as in the proof of Theorem 4.6. We again consider $g \in \mathcal{G}^+_\lambda$ for some $\lambda > 0$, and prove that $\xi \cdot \mathbb{P}$ is locally bounded for arbitrary $\xi \in \mathbb{R}^{3+1} \oplus \mathfrak{g}$.

Note that $\mathring{\nabla}\hat{\xi}$ has compact support, so the second term of both equations (5.17) and (5.18), and the third and fourth terms in (5.16) are all clearly bound.

From Lemma 5.2, it can be seen that the dominant part of the remaining terms in (5.16) agree with the scalar curvature of g, which can be shown to be integrable from the assumption $s \in L^1$. First note

$$\|\hat{\xi}^0_\infty(\mathring{\nabla}^i\mathring{\nabla}^j g_{ij} - \mathring{\Delta}\mathrm{tr}_{\mathring{g}} g)\|_{1,-3} \leq \|\hat{\xi}^0_\infty\|_{\infty,0}\|\mathring{\nabla}^i\mathring{\nabla}^j g_{ij} - \mathring{\Delta}\mathrm{tr}_{\mathring{g}} g)\|_{1,-3},$$

then from Lemma 5.2 we have

$$\|\mathring{\nabla}^i\mathring{\nabla}^j g_{ij} - \mathring{\Delta}\mathrm{tr}_{\mathring{g}} g)\|_{1,-3} \leq c(\lambda)(1 + \|g - \mathring{g}\|^2_{2,2,-1/2}) + \|R\|_{1,-3}.$$

From the Hamiltonian constraint (4.13), we have

$$\|R\|_{1,-3} \leq c(\lambda)(\|s\|_{1,-3} + \|\pi^2\|_{1,-3}) \leq c(\lambda)(\|s\|_{3,-1} + \|\pi\|^2_{1,2,-3/2}).$$

Putting all of this together, we conclude that $\xi^0_\infty \mathbb{P}_0$ is locally bounded.

The first term in (5.17) is estimated by the momentum constraint (4.14) and again makes use of the condition $s \in L^1$. We have

$$\begin{aligned}
\|\xi^i_\infty \nabla_j \pi^j_i\|_{1,-3} &\leq c\|\xi_\infty\|_{\infty,0}(\|s\|_{1,-3} + \|\varepsilon\mathring{\nabla}A\|_{1,-3} + \|A\mathring{\nabla}\varepsilon\|_{1,-3}) \\
&\leq c\|\xi_\infty\|_{\infty,0}(\|s\|_{1,-3} + \|\varepsilon\|_{2,-3/2}\|\mathring{\nabla}A\|_{2,-3/2} + \|A\|_{2,-1/2}\|\mathring{\nabla}\varepsilon\|_{2,-5/2}) \\
&\leq c\|\xi_\infty\|_{\infty,0}(\|s\|_{1,-3} + \|\varepsilon\|_{1,2,-3/2}\|A\|_{1,2,-1/2}),
\end{aligned}$$

and then we have

$$\begin{aligned}
\|\xi^i_\infty \mathring{\nabla}_j \pi^j_i\|_{1,-3} &\leq \|\xi^i_\infty \nabla_j \pi^j_i\|_{1,-3} + \|\hat{\xi}_\infty\|_{\infty,0}\|\tilde{\Gamma}\pi\|_{1,-3} \\
&\leq \|\xi^i_\infty \nabla_j \pi^j_i\|_{1,-3} + c(\lambda)\|\hat{\xi}_\infty\|_{\infty,0}(1 + \|\mathring{\nabla}g\|^2_{1,2-3/2})\|\pi\|_{2,-3/2},
\end{aligned}$$

where we have made use of Proposition 4.4.

Finally, the remaining term in (5.18) is estimated similarly:

$$\begin{aligned}\|\hat{\xi}^a_\infty \mathring{\nabla}_i E^i_a\|_{1,-3} \le& \|\hat{\xi}^a\|_{\infty,0}\|\mathring{\nabla}_i E^i_a\|_{1,-3}\\ \le& c\|\hat{\xi}^a\|_{\infty,0}(\|s\|_{1,-3}+\|[E,A]\|_{1,-3})\\ \le& c\|\hat{\xi}^a\|_{\infty,0}(\|s\|_{1,-3}+\|E\|_{2,-3/2}\|A_{\mathfrak{k}}\|_{2,-3/2}).\end{aligned}$$

It follows that $\mathbb{P}$ is smooth. □

Note that the above theorem is valid in both cases, (a) and (b), as the weighted Hölder and Sobolev inequalities are valid on $\mathcal{M}_0$ (Remark 3.5). However, in case (b) we are unable to directly identify $\mathbb{P}$ with (E, p_i, Q_a). We next seek to construct similar volume integrals to represent the physical quantities in case (b).

Let $\hat{\xi}_\Sigma \in C^\infty(\Lambda^0 \times T\mathcal{M} \times \mathfrak{g} \otimes \Lambda^0)$ be supported on a neighbourhood of Σ, and define $\hat{\xi}_{\text{ref}} := \hat{\xi}_\infty + \hat{\xi}_\Sigma$. This permits us to prescribe values of $\hat{\xi}_{\text{ref}}$ both near infinity and on Σ. Analogously to the spaces $W^{k,p}_{\xi_\infty}$, we define

$$L^p_{\hat{\xi}_{\text{ref}}} := \left\{\xi : \xi - \hat{\xi}_{\text{ref}} \in L^p_{-1/2}\big(\Lambda^0(\mathcal{M}_0) \times T\mathcal{M}_0 \times \mathfrak{g} \otimes \Lambda^0(\mathcal{M}_0)\big)\right\}, \qquad (5.20)$$

$$W^{k,p}_{\hat{\xi}_{\text{ref}}} := \left\{\xi : \xi - \hat{\xi}_{\text{ref}} \in \mathring{W}^{k,p}_{-1/2}\big(\Lambda^0(\mathcal{M}_0) \times T\mathcal{M}_0 \times \mathfrak{g} \otimes \Lambda^0(\mathcal{M}_0)\big)\right\}, \qquad (5.21)$$

recalling that the notation, $\mathring{W}^{k,p}_{-1/2}$, refers to the completion of C^∞_c with respect to the $W^{k,p}_{-1/2}$ norm. The $W^{k,p}_{\hat{\xi}_{\text{ref}}}$ spaces then include our prescribed asymptotics and boundary conditions in the trace sense, while the $L^p_{\hat{\xi}_{\text{ref}}}$ spaces only include the asymptotics.

Non-zero values of $\hat{\xi}_\Sigma$ mean that we can no longer write the surface integrals at infinity as volume integrals of total divergences, as we will also have contributions on Σ. We are only interested in the case when $\hat{\xi}^0_\Sigma = 0$, and in this case we write

$$16\pi\xi^0_\infty \mathbb{P}'_0(g) = \int_{\mathcal{M}_0} \Big(\xi^0_{\text{ref}}(\mathring{g}^{ki}\mathring{g}^{jl}\mathring{\nabla}_k\mathring{\nabla}_l g_{ij} - \mathring{\Delta}\text{tr}_{\mathring{g}}g) + \mathring{g}^{ki}\mathring{g}^{jl}\mathring{\nabla}_k\xi^0_{\text{ref}}(\mathring{\nabla}_l g_{ij} - \mathring{\nabla}_i\mathring{g}_{jl})\Big)\sqrt{\mathring{g}}, \tag{5.22}$$

$$16\pi\xi^i_\infty \mathbb{P}'_i(\pi) = \int_{\mathcal{M}_0} \Big(2\xi^i_{\text{ref}}\mathring{\nabla}_j\pi^j_i + 2\pi^j_i\mathring{\nabla}_j\xi^i_{\text{ref}} + \mathring{\nabla}_i(\varepsilon^i_a A^a_j)\xi^j_{\text{ref}} + \varepsilon^i_a A^a_j\mathring{\nabla}_i\xi^j_{\text{ref}}\Big) + \oint_\Sigma \Big(2\hat{\xi}^i_{\text{ref}}\pi^j_i - \varepsilon^j_a A^a_i\hat{\xi}^i_{\text{ref}}\Big)\,dS_j, \tag{5.23}$$

$$16\pi\xi^a_\infty \mathbb{P}'_a(\varepsilon) = 4\int_{\mathcal{M}_0} \Big(\hat{\xi}^a_\infty\mathring{\nabla}_i E^i_a + E^i_a\mathring{\nabla}_i\hat{\xi}^a_\infty\Big) - 4\oint_\Sigma \hat{\xi}^a_{\text{ref}}E^i_a dS_i. \tag{5.24}$$

While (5.23) contains the terms (g, A, ε), the quantity $\mathbb{P}'_i$ only depends on π. This can be seen by using the divergence theorem to write the bulk integral as surface integrals at infinity and on Σ, and then noting that the terms on Σ cancel. The terms at infinity correspond to the ADM momentum, and some Yang-Mills terms that can be shown to vanish by Proposition 3.8:

$$\begin{aligned}\|\varepsilon A\hat{\xi}_{\text{ref}}\|_{L^1(S_R)} \leq& cR^{1/2}\|\hat{\xi}_{\text{ref}}\|_{\infty:S_R}\|A\|_{\infty:S_R}\|\varepsilon\|_{1,2,-3/2:A_R}\\ \leq& R^{1/2}o(R^{-1/2})\|\varepsilon\|_{1,2,-3/2},\end{aligned}$$

which goes to zero as R goes to infinity. In fact, provided the boundary integrals at infinity arising from the divergence theorem are well-defined, then $\mathbb{P}'$ agrees with $\mathbb{P}$ and (E, p_i, V) (Theorem 5.6).

Note that the surface integral on Σ in (5.23) corresponds exactly to $\tilde{J}_{\hat{\xi}_{\text{ref}}}$, and if $\hat{\xi}^a_{\text{ref}} = \xi^a_\Sigma \in \mathfrak{g}$ is constant on Σ, then the surface integral in (5.24) corresponds exactly to $\xi^a_\Sigma Q_{\Sigma\,a}$. We therefore have the following immediate corollary of Proposition 5.1 and Theorem 5.3.

Corollary 5.4. *If $s \in L^1$, $\hat{\xi}^a_{\text{ref}} = \xi^a_\Sigma \in \mathfrak{g}$ is constant and $\xi \in W^{2,2}_{\hat{\xi}_{\text{ref}}}$, then $\mathbb{P}' : \mathcal{C}(s) \to \mathbb{R}^{3,1} \oplus \mathfrak{g}$ is a smooth map.*

We now show that the the definitions of $\mathbb{P}$ and $\mathbb{P}'$ indeed agree with the associated surface integrals at infinity. For this, we employ the following Lemma from [11]:

Lemma 5.5 (Lemma 4.3 from [11]). *Suppose $\mathcal{M} = \bigcup_{k\geq} \mathcal{M}_k$ is an exhaustion of a non-compact n-dimensional manifold $\mathcal{M}$ by compact subsets with smooth boundaries $\partial\mathcal{M}_k$ and suppose $\beta \in W^{1,2}_{loc}(\Lambda^{n-1}(\mathcal{M}))$ satisfies $d\beta \in L^1(\Lambda^n(\mathcal{M}))$. Then*

$$\oint_{\partial\mathcal{M}_\infty} \beta := \lim_{k\to\infty} \oint_{\partial\mathcal{M}_k} \beta \qquad \textit{exists.}$$

Theorem 5.6. *Suppose $(g, A, \pi, \varepsilon) \in \mathcal{C}(s)$, $s \in L^1$, then the surface integrals at infinity giving (E, p, V) are well-defined and we have $(E, p, V) = \mathbb{P} = \mathbb{P}'$.*

Proof. The integrands in (5.16)-(5.18) have been shown to be L^1 by Theorem 5.3, and the related surface integrands are clearly seen to be $W^{1,2}_{loc}$. In light of both the discussion above and Lemma 5.5, the equivalence is clear.

$\square$

Chapter 6

The First Law of Black Hole Mechanics as a Condition for Stationarity

It seems plain and self-evident, yet it needs to be said: the isolated knowledge obtained by a group of specialists in a narrow field has in itself no value whatsoever, but only in its synthesis with all the rest of knowledge and only inasmuch as it really contributes in this synthesis toward answering the demand, "Who are we?".

Erwin Schrödinger

We are now in a position to discuss the relationship between the first law of black hole mechanics and stationary initial data. It is mentioned in the introduction, that this relationship was considered by Sudarsky and Wald in 1992 [57], however a rigorous proof was not established. We consider separately, the cases (a) and (b) of Chapter 5, in Sections 6.2 and 6.3 respectively. In both cases, the differential relationship given by the first law is shown to give a condition for stationarity. In case (b), we also give evidence suggesting that the boundary surface is indeed a horizon if the first law holds there. In fact, if the first law holds and the solution is axially symmetric, then we conclude that the boundary surface is the bifurcation surface of a bifurcate Killing horizon.

In case (a), when there is no interior boundary, the condition for stationarity is given by

$$dm + V \cdot dQ = 0, \tag{6.1}$$

where m is the mass, V is the Yang-Mills electric potential and Q is the Yang-Mills electric charge measured at infinity. Note that the condition (6.1) is not exactly the first law, it is however related. If we restrict ourselves to trivial topology[1] and since we consider only a single end, then there will be no horizon and therefore no black hole area, charge or angular momentum. However we argue that the electric charge at infinity should also be considered and will have the opposite sign to the usual expression for the first law, where the electric charge is considered at the horizon.

In the pure Einstein case, this condition further reduces to $dm = 0$; stationary solutions are equated with critical points of the mass. This was first argued non-rigorously by Brill, Deser and Fadeev [17] in 1968 and rigorously established by Bartnik in 2005 [11], as critical points of the ADM mass relate to the Bartnik quasilocal mass [10]. For this reason, we follow the ideas of Bartnik to establish it the cases presented here.

In both cases, we need to modify the usual ADM Hamiltonian, so we first digress briefly to discuss Hamiltonians in general relativity.

6.1 Hamiltonians

Before considering the Hamiltonian for the Einstein-Yang-Mills equations, we first examine the usual ADM Hamiltonian for general relativity. Recall now, the Einstein constraints, evolution equations and canonical Hamiltonian variables from

[1] Recently Klinkhamer [45] has given nonsingular vacuum and electrovac black hole solutions, which are Schwarzschild and Reissner Nördstrom respectively, outside the horizon. Rather than a singularity, the manifold has unusual topology inside the horizon, however it is not clear exactly what role these solutions play in this picture.

Section 2.1. The ADM Hamiltonian, from which these equations can be derived, is given by

$$\mathcal{H}_{ADM}^{(N,X)}(g,\pi) = -\int_{\mathcal{M}} \tilde{\Phi}(g,\pi)\cdot(N,X), \tag{6.2}$$

where $\tilde{\Phi}(g,\pi) = (\Phi_0(g,0,\pi,0), \Phi_i(g,0,\pi,0))$ is the constraint map in the pure gravity case. The superscript (N,X) indicates the Hamiltonian's dependence on (N,X), however we will generally consider this to be fixed. Notice that the Hamiltonian vanishes when the constraints are satisfied; as such, this is sometimes called the pure constraint form of the Hamiltonian. The variational derivative of the Hamiltonian density is then simply

$$DH_{(g,\pi)}^{(N,X)}[h,p] = -(N,X)\cdot D\tilde{\Phi}_{(g,\pi)}[h,p]. \tag{6.3}$$

Formally integrating by parts gives

$$DH_{(g,\pi)}^{(N,X)} = -D\tilde{\Phi}^*_{(g,\pi)}[N,X]. \tag{6.4}$$

Hamilton's equations then give the correct evolution equations:

$$\frac{\partial}{\partial t}\begin{bmatrix} g \\ \pi \end{bmatrix} = -J\circ D\tilde{\Phi}^*_{(g,\pi)}[N,X], \tag{6.5}$$

where

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{6.6}$$

and t is the flow parameter of (N,X), interpreted as a timelike four-vector field in the full spacetime. This can be seen by comparing Hamilton's equations, (6.4, 6.5), with the Einstein evolution equations, (2.4, 2.5). The four-vector, (N,X), is to be interpreted as the lapse-shift vector from Section 2.1.

Equation (6.5) motivates a result by Moncrief, equating non-trivial (N,X) in the kernel of $D\tilde{\Phi}_{(g,\pi)}$, with Killing vectors in the full spacetime [48]. If $D\tilde{\Phi}^*_{(g,\pi)}[N,X] = 0$, for some (N, X) that corresponds to a vector field in the spacetime that is time-like at infinity, then we say (g, π) is a stationary initial data set for the Einstein equations.

In the asymptotically flat case, the integration by parts above will result in a surface integral at infinity that does not vanish in general. This surface term corresponds to the variational derivative of the ADM mass, so in order to generate the correct equations of motion, the mass is added to the Hamiltonian. This was established by Regge and Teitelboim [50] and the modified Hamiltonian now bears their names. We will discuss the Regge-Teitelboim Hamiltonian in more detail in the subsequent sections.

Now we move to discuss the the Hamiltonian for the coupled system, which behaves analogously. Above, the lapse and shift act as the Lagrange multipliers corresponding to the Hamiltonian and momentum constraints respectively. In the coupled system, the electric potential is the Lagrange multiplier for the Gauss constraint, and the pure constraint form of the Einstein-Yang-Mills Hamiltonian is given by

$$\mathcal{H}^{\xi}_{EYM}(g, A, \pi, \varepsilon) = -\int_{\mathcal{M}} \Phi(g, A, \pi, \varepsilon) \cdot \xi. \tag{6.7}$$

Differentiating the Hamiltonian density and formally integrating by parts reveals that Hamilton's equations again give the correct equations of motion:

$$\frac{\partial}{\partial t}\begin{bmatrix} g \\ A \\ \pi \\ \varepsilon \end{bmatrix} = -J \circ D\Phi^{*}_{(g,A,\pi,\varepsilon)}[\xi], \tag{6.8}$$

where

$$J = \begin{bmatrix} 0 & Id \\ -Id & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}. \tag{6.9}$$

The evolution equations for (g, π) now include the Yang-Mills stress energy tensor and the two additional equations give the evolution of (A, ε) (cf. (2.15), (2.16)). Where previously t was the flow parameter for a vector field on ${}^4V$, here it is the flow parameter of a vector field on the bundle, P. That is, the evolution is interpreted as a simultaneous time evolution and continuous gauge transformation.

As above, if $D\Phi^*[\xi] = 0$ with (N, X) timelike at infinity, then we say (g, A, π, ε) is a stationary initial data set for the Einstein-Yang-Mills equations. In this case, V corresponds to a gauge choice that ensures $\frac{\partial E}{\partial t} = 0 = \frac{\partial A}{\partial t}$. The pure constraint Einstein-Yang-Mills Hamiltonian also requires a modification to cancel out the boundary terms arising from the integration by parts, and therefore give the correct equations of motion. However, we will reserve discussion on this modification until the next Section.

6.2 Without an Interior Boundary

This Section considers case (a) of the preceding Chapter, where $\mathcal{M}$ has a single asymptotic end and no interior boundary. For this, we directly make use of the definitions and constructions of Chapter 4.

We begin by establishing that the pure constraint Einstein-Yang-Mills Hamiltonian is a smooth map on the phase space.

Proposition 6.1. *The map $\mathcal{H}_{EYM} : \mathcal{F} \times \mathcal{N}^* \to \mathbb{R}$ is smooth in the sense of Fréchet differentiability.*

Proof. Smoothness with respect to (g, A, π, ε) follows directly from the smoothness of Φ (Theorem 4.6). By the weighted Hölder's inequality,

$$|\mathcal{H}^{\xi}_{EYM}(g, A, \pi, \varepsilon)| = \|\xi \cdot \Phi\|_{1,-3} \leq \|\xi\|_{2,-1/2}\|\Phi\|_{2,-5/2}, \tag{6.10}$$

and since $\mathcal{H}_{EYM}$ is bounded and linear in ξ, smoothness follows. □

We will now prove that the formal adjoint of the linearised constraint map is indeed equal to the true adjoint for $\xi \in W^{2,2}_{-1/2}$. That is, if the evolution vector vanishes at infinity, then $\mathcal{H}_{EYM}$ gives the correct equations of motion. However as mentioned above, this is not the case when we permit ξ to be a constant translation near infinity.

Theorem 6.2. *Suppose $\xi \in W^{2,2}_{-1/2}$, then*

$$D\mathcal{H}^{\xi}_{EYM} \cdot (h, b, p, f) = -\int_{\mathcal{M}} D\Phi^*[\xi] \cdot (h, b, p, f), \tag{6.11}$$

for all $(h, b, p, f) \in T_{(g,A,\pi,\varepsilon)}\mathcal{F}$.

Proof. To arrive at the expression for $D\Phi^*$, boundary terms arising from integration by parts were disregarded. To establish (6.11), we must demonstrate that these boundary terms at infinity do indeed vanish. We consider the difference between $\xi \cdot D\Phi[h, b, p, f]$ and $(h, b, p, f) \cdot D\Phi^*[\xi]$ (see (4.20)-(4.26)), and after canceling out common terms, we arrive at the following expression:

$$\begin{aligned}
&(h, b, p, f) \cdot D\Phi^*[\xi] - \xi \cdot D\Phi[h, b, p, f] = \qquad (6.12)\\
&\quad \nabla^i\Big((N(\mathring{\nabla}_i \mathrm{tr}_g h - \nabla^j h_{ij}) + \mathring{\nabla}^j(N)h_{ij} - \mathrm{tr}_g h \mathring{\nabla}_i(N))\sqrt{g} - 2X^j p_{ij} + V^a f_{ai}\Big)\\
&\quad - \nabla^i\Big(2\pi^k_i h_{jk} X^j - \pi^{jk} h_{jk} X_i + \epsilon_{ijk} b^{ak} B^j_a N\sqrt{g} + \varepsilon_{ia} b^a_j X^j - X_i \varepsilon^j_a b^a_j + f_{ai} X^j A^a_j\Big).
\end{aligned}$$

The boundary terms have been expressed as two separate divergences, corresponding to their decay rates at infinity — a distinction that will be important later.

For the sake of presentation, we define

$$\mathcal{B}^1_i = (N(\mathring{\nabla}_i \mathrm{tr}_g h - \nabla^j h_{ij}) + \mathring{\nabla}^j(N)h_{ij} - \mathrm{tr}_g h \mathring{\nabla}_i(N))\sqrt{g} - 2X^j p_{ij} + V^a f_{ai},$$
$$\mathcal{B}^2_i = 2\pi^k_i h_{jk} X^j - \pi^{jk} h_{jk} X_i + \epsilon_{ijk} b^{ak} B^j_a N\sqrt{g} + \varepsilon_{ia} b^a_j X^j - X_i \varepsilon^j_a b^a_j + f_{ai} X^j A^a_j,$$

so that (6.12) can now be written as

$$(h, b, p, f) \cdot D\Phi^*[\xi] - \xi \cdot D\Phi[h, b, p, f] = \nabla^i \mathcal{B}^1_i + \nabla^i \mathcal{B}^2_i. \tag{6.13}$$

The boundary terms at infinity are then given by $\lim_{R\to\infty} \oint_{S_R} (\mathcal{B}^1_i + \mathcal{B}^2_i) dS^i$, so we must show that this vanishes to complete the proof. First, we demonstrate that these surface integrals are well-defined in the trace sense, and the limit as R tends to infinity is also well-defined. In order to do this, we show that the hypotheses of Lemma 5.5 are satisfied. By construction, each of $(h, b, p, f) \cdot D\Phi^*[\xi]$ and $\xi \cdot D\Phi[h, b, p, f]$ are integrable, so all that remains to be shown is $\mathcal{B}^1, \mathcal{B}^2 \in W^{1,2}_{loc}$. The weighted Poincaré inequality gives

$$\|\mathcal{B}^1\|_{1,2,-2} \le c\|\mathring{\nabla}\mathcal{B}^1\|_{2,-3},$$
$$\|\mathcal{B}^2\|_{1,2,-2} \le c\|\mathring{\nabla}\mathcal{B}^2\|_{2,-3},$$

and the usual weighted Sobolev-type inequalities are used below to obtain the required estimates. Recall that g is Hölder continuous and bounded on $\mathcal{M}$, so we need not consider the raising and lowering of indices in the estimates to follow.

The usual weighted inequalities give

$$\begin{aligned}
\|\mathring{\nabla}\mathcal{B}^1\|_{2,-3} \le & \, c\big(\|\mathring{\nabla}N\|_{4,-3/2}\|\mathring{\nabla}h\|_{4,-3/2} + \|N\|_{\infty,-1/2}\|\mathring{\nabla}^2 h\|_{2,-5/2} \\
& + \|\mathring{\nabla}X\|_{4,-3/2}\|p\|_{4,-3/2} + \|X\|_{\infty,-1/2}\|\mathring{\nabla}p\|_{2,-5/2} \\
& + \|\mathring{\nabla}f\|_{2,-5/2}\|V\|_{\infty,-1/2} + \|\mathring{\nabla}V\|_{4,-3/2}\|f\|_{4,-3/2} \\
& + \|\mathring{\nabla}^2 N\|_{2,-5/2}\|h\|_{\infty,-1/2}\big) \\
\le & \, c\big(\|\mathring{\nabla}N\|_{1,2,-3/2}\|\mathring{\nabla}h\|_{1,2,-3/2} + \|N\|_{2,2,-1/2}\|\mathring{\nabla}^2 h\|_{2,-5/2} \\
& + \|\mathring{\nabla}X\|_{1,2,-3/2}\|p\|_{1,2,-3/2} + \|X\|_{2,2,-1/2}\|\mathring{\nabla}p\|_{2,-5/2} \\
& + \|\mathring{\nabla}f\|_{2,-5/2}\|V\|_{2,2,-1/2} + \|\mathring{\nabla}V\|_{1,2,-3/2}\|f\|_{1,2,-3/2} \\
& + \|\mathring{\nabla}^2 N\|_{2,-5/2}\|h\|_{2,2,-1/2}\big) \\
\le & \, c\|\xi\|_{2,2,-1/2}\big(\|h\|_{2,2,-1/2} + \|p\|_{1,2,-3/2} + \|f\|_{1,2,-3/2}\big).
\end{aligned}$$

For the sake of presentation, we again group terms with the same regularity and decay rate; we define $\alpha = (h, b, A) \in W^{2,2}_{-1/2}$ and $\beta = (\pi, B, \varepsilon, f) \in W^{1,2}_{-3/2}$. We can then write $\mathcal{B}^2$ as the collection of terms $\mathcal{B}^2 \sim \alpha\beta\xi$ and the estimate can be concisely written as follows:

$$\begin{aligned}
\|\mathring{\nabla}\mathcal{B}^2\|_{2,-3} \le & \, c\big(\|\alpha\beta\mathring{\nabla}\xi\|_{2,-3} + \|\xi\alpha\mathring{\nabla}\beta\|_{2,-3} + \|\beta\xi\mathring{\nabla}\alpha\|_{2,-3}\big) \\
\le & \, c\big(\|\alpha\|_{\infty,-1/2}\|\beta\|_{4,-3/2}\|\mathring{\nabla}\xi\|_{4,-1} + \|\xi\|_{\infty,0}\|\alpha\|_{\infty,-1/2}\|\mathring{\nabla}\beta\|_{2,-5/2} \\
& + \|\beta\|_{4,-3/2}\|\xi\|_{\infty,0}\|\mathring{\nabla}\alpha\|_{4,-3/2}\big) \\
\le & \, c\big(\|\alpha\|_{2,2,-1/2}\|\beta\|_{1,2,-3/2}\|\mathring{\nabla}\xi\|_{1,2,-1} + \|\xi\|_{2,2,0}\|\alpha\|_{2,2,-1/2}\|\mathring{\nabla}\beta\|_{2,-5/2} \\
& + \|\beta\|_{1,2,-3/2}\|\xi\|_{2,2,0}\|\mathring{\nabla}\alpha\|_{1,2,-3/2}\big) \\
\le & \, c\|\alpha\|_{2,2,-1/2}\|\beta\|_{1,2,-3/2}\|\xi\|_{2,2,0}.
\end{aligned}$$

Since we have $\mathcal{B}^1, \mathcal{B}^2 \in W^{1,2}_{loc}$, Lemma 5.5 tells us that the surface integrals at infinity are well-defined.

We again make use of the usual weighted Sobolev-type inequalities and Proposition 3.8 to obtain estimates on S_R:

$$\begin{aligned}\oint_{S_R} \mathcal{B}^1_i dS^i &\leq c\big(\|\xi\|_{\infty:S_R}(\|\mathring{\nabla} h\|_{1:S_R} + \|p\|_{1:S_R} + \|f\|_{1:S_R}) + \|h\|_{\infty:S_R}\|\mathring{\nabla} N\|_{1:S_R}\big) \\ &\leq cR^{1/2}o(R^{-1/2})(\|\mathring{\nabla} h\|_{1,2,-3/2} + \|p\|_{1,2,-3/2} + \|f\|_{1,2,-3/2} + \|\mathring{\nabla} N\|_{1,2,-3/2}) \\ &= o(1),\end{aligned}$$

where $o(1)$ here refers to the asymptotic behaviour as $R \to \infty$. Note that we have made use of the Hölder continuity of ξ obtained by the Morrey-Sobolev embedding, $C^{0,1/2} \subset W^{2,2}$, to write $\|\xi\|_{\infty:S_R} = o(R^{-1/2})$ for $\xi \in W^{2,2}_{-1/2}$. Similarly, we obtain

$$\begin{aligned}\oint_{S_R} \mathcal{B}^2_i dS^i &\leq c\|\alpha\|_{\infty:S_R}\|\xi\|_{\infty:S_R}\|\beta\|_{1:S_R} \\ &\leq o(1)\|\xi\|_{\infty:S_R}\|\beta\|_{1,2,-3/2} \\ &= o(R^{-1/2}).\end{aligned}$$

It follows that the boundary terms at infinity vanish and therefore we have

$$\int_{\mathcal{M}} \xi \cdot D\Phi[h,b,p,f] = \int_{\mathcal{M}} (h,b,p,f) \cdot D\Phi^*[\xi].$$

□

Note that the difference in the decay rates of $\mathcal{B}^1$ and $\mathcal{B}^2$ at infinity are evident in the above estimates; to ensure $\lim_{R\to\infty} \oint_{S_R} \mathcal{B}^1_i dS^i = 0$, we require $\xi = o(r^{-1/2})$, however we only need ξ to be bounded to ensure $\lim_{R\to\infty} \oint_{S_R} \mathcal{B}^2_i dS^i = 0$.

Recall that ξ will correspond to a time evolution vector field, so we would like the four-vector component to be asymptotic to a constant timelike vector in ${}^4V$. In which case, the boundary terms from $\mathcal{B}^2$ still vanish, but those from $\mathcal{B}^1$ are not even necessarily finite. However, when they are finite these terms do have an important physical interpretation. The boundary terms from $\mathcal{B}^1$ correspond to the first variation of the ADM energy-momentum and electric charge. Not only are these boundary terms non-vanishing when ξ is permitted to be an asymptotic translation, but the Hamiltonian itself is no longer well defined. The condition that $\xi = o(r^{-1/2})$ is required to ensure that the integral in (6.7) is convergent.

This motivates the introduction of a new Hamiltonian à la Regge-Teitelboim. For $\xi \in L^2_{\xi_\infty}$, an asymptotic translation (see (5.14)), we define

$$\mathcal{H}^{\xi}_{RT}(g, A, \pi, \varepsilon) := 16\pi\xi_\infty \cdot \mathbb{P}(g, A, \pi, \varepsilon) - \int_{\mathcal{M}} \xi \cdot \Phi(g, A, \pi, \varepsilon), \tag{6.14}$$

where $\mathbb{P}$ is defined by (5.16) - (5.18). When the source-free constraints are satisfied, this Hamiltonian is identically $16\pi\xi_\infty \cdot \mathbb{P}(g, A, \pi, \varepsilon)$ and therefore gives some notion of a total energy for the system. While neither of the terms in (6.14) are well-defined on the entire phase space, it will be shown that when we combine these integrals the dominant terms cancel out, and the resulting expression is well-defined on all of $\mathcal{F}$. Writing (6.14) as an integral over $\mathcal{M}$, we arrive at the regularised Hamiltonian:

$$\begin{aligned}\mathcal{H}^{\xi}(g, A, \pi, \varepsilon) = &\int_{\mathcal{M}} (\hat{\xi}_\infty - \xi) \cdot \Phi + \int_{\mathcal{M}} \hat{\xi}^0_\infty \big(\mathring{g}^{ik}\mathring{g}^{jl}(\mathring{\nabla}_k\mathring{\nabla}_l g_{ij} - \mathring{\nabla}_i\mathring{\nabla}_k(g_{jl})\sqrt{\mathring{g}}) - \Phi_0\big) \\ &+ \int_{\mathcal{M}} \mathring{g}^{ik}\mathring{g}^{jl}\mathring{\nabla}_k\hat{\xi}^0_\infty(\mathring{\nabla}_l g_{ij} - \mathring{\nabla}_i g_{jl})\sqrt{\mathring{g}} + \int_{\mathcal{M}} \hat{\xi}^i_\infty(2\mathring{\nabla}_j\pi^j_i - \Phi_i) \\ &+ \int_{\mathcal{M}} 2\pi^j_i\mathring{\nabla}_j\hat{\xi}^i_\infty + \int_{\mathcal{M}} \hat{\xi}^a_\infty(4\mathring{\nabla}_i E^i_a - \Phi_a) + 4\int_{\mathcal{M}} E^i_a\mathring{\nabla}_i\hat{\xi}^a_\infty.\end{aligned} \tag{6.15}$$

The regularised Hamiltonian has been expressed as seven separate integrals, as each of these integrals can be shown to converge independently.

Theorem 6.3. *The regularised Hamiltonian, defined by (6.15), is a smooth functional on $\mathcal{F} \times L^2_{\xi_\infty}$. Furthermore, if $\xi \in W^{2,2}_{\xi_\infty}$ then for all $(g, A, \pi, \varepsilon) \in \mathcal{F}$ and $(h, b, p, f) \in T_{(g,A,\pi,\varepsilon)}\mathcal{F}$, we have*

$$D\mathcal{H}^{\xi}[h, b, p, f] = -\int_{\mathcal{M}} (h, b, p, f) \cdot D\Phi^*[\xi]. \tag{6.16}$$

That is, the regularised Hamiltonian gives the correct equations of motion (cf. 6.8).

Proof. Again, we simply must show that the Hamiltonian is locally bounded, then smoothness follows by the exact same argument as that used in the proof of Theorem 4.6. The first integral is bounded by $\|(\hat{\xi}_\infty - \xi)\|_{2,-1/2}\|\Phi\|_{2,-5/2}$, and the third, fifth and seventh integrals are bounded since $\mathring{\nabla}\hat{\xi}_\infty$ is compactly supported. Suppose $g \in \mathcal{G}^+_\lambda$, then Lemma 5.2 gives a bound for the second integral:

$$\begin{aligned}\int_{\mathcal{M}} \hat{\xi}^0_\infty \big(\mathring{g}^{ik}\mathring{g}^{jl}(\mathring{\nabla}_k\mathring{\nabla}_l g_{ij} - \mathring{\nabla}_i\mathring{\nabla}_k(g_{jl})\sqrt{\mathring{g}}) - \Phi_0\big)&\\ \leq c(\lambda)\,\|\hat{\xi}_\infty\|_{\infty,0}(1 + \|g-\mathring{g}\|^2_{2,2,-1/2} + \|\pi^2\|_{2,-3})&\\ \leq c(\lambda)\,\|\hat{\xi}_\infty\|_{\infty,0}(1 + \|g-\mathring{g}\|^2_{2,2,-1/2} + \|\pi^2\|_{1,2,-3/2}).&\end{aligned}$$

The fourth and sixth integrals are then bounded similarly:

$$\begin{aligned}\int_{\mathcal{M}} \hat{\xi}^i_\infty(2\mathring{\nabla}_j\pi^j_i - \Phi_i) &\leq c\,\|\hat{\xi}_\infty\|_{\infty,0}(\|\tilde{\Gamma}\pi\|_{1,-3} + \|\varepsilon\mathring{\nabla}A\|_{1,-3} + \|A\mathring{\nabla}\varepsilon\|_{1,-3})\\ &\leq c\,\|\hat{\xi}_\infty\|_{\infty,0}(\|\tilde{\Gamma}\|_{2,-3/2}\|\pi\|_{2,-3/2} + \|\varepsilon\|_{2,-3/2}\|\mathring{\nabla}A\|_{2,-3/2}\\ &\qquad + \|A\|_{2,-1/2}\|\mathring{\nabla}\varepsilon\|_{2,-5/2}),\end{aligned}$$

$$\begin{aligned}\int_{\mathcal{M}} \hat{\xi}^a_\infty(4\mathring{\nabla}_i E^i_a - \Phi_a) &\leq c\|\hat{\xi}_\infty\|_{\infty,0}\|A_{\mathfrak{k}}\varepsilon\|_{1,-3}\\ &\leq c\|\hat{\xi}_\infty\|_{\infty,0}\|A_{\mathfrak{k}}\|_{2,-3/2}\|\varepsilon\|_{2,-3/2}.\end{aligned}$$

Note that $\hat{\xi}$ is smooth and bounded, so it follows that $\|\hat{\xi}\|_{\infty,0}$ is bounded. Since the regularised Hamiltonian is locally bounded, smoothness follows.

Next we establish the validity of (6.16), by again considering the seven integrals in (6.15) separately. Since $\hat{\xi}_\infty - \xi \in W^{2,2}_{-1/2}$, Theorem (6.2) allows us to write the first integral as

$$\int_{\mathcal{M}} (h,b,p,f)\cdot D\Phi^*[\hat{\xi}_\infty - \xi]. \tag{6.17}$$

The variation of the second and third terms combine to give

$$\int_{\mathcal{M}} \Big\{ \mathring{g}^{ik}\mathring{\nabla}_k(\hat{\xi}^0_\infty \mathring{g}^{jl}(\mathring{\nabla}_l h_{ij} - \mathring{\nabla}_i h_{jl}))\sqrt{\mathring{g}} - \nabla^i(\hat{\xi}^0_\infty(\nabla^j h_{ij} - \nabla_i \operatorname{tr}_g h))\sqrt{g} \\ + \nabla^i(h_{ij}\nabla^j\hat{\xi}^0_\infty - \operatorname{tr}_g h\nabla_i\hat{\xi}^0_\infty)\sqrt{g} - (h,b,p,f)\cdot D\Phi^*[\hat{\xi}^0_\infty]\Big\}, \tag{6.18}$$

where the two middle terms in (6.18) arise from the difference,

$$(h,b,p,f)\cdot D\Phi^*[\hat{\xi}^0_\infty] - \hat{\xi}^0_\infty \cdot D\Phi[h,b,p,f]; \qquad \text{see (6.12)}. \tag{6.19}$$

Note that the third divergence in (6.18) is compactly supported, so it follows that its integral vanishes. Next consider the second term in (6.18):

$$\begin{aligned} \nabla^i(\hat{\xi}^0_\infty(\nabla^j h_{ij} - \nabla_i \operatorname{tr}_g h))\sqrt{g} &= \mathring{\nabla}_k(g^{ik}\hat{\xi}^0_\infty g^{jl}(\nabla_l h_{ij} - \nabla_i h_{jl})(\sqrt{g} - \sqrt{\mathring{g}})) \\ &\quad \mathring{\nabla}_k(g^{ik}\hat{\xi}^0_\infty g^{jl}(\nabla_l h_{ij} - \nabla_i h_{jl}))\sqrt{\mathring{g}} \\ &= \mathring{\nabla}_k(g^{ik}\hat{\xi}^0_\infty g^{jl}(\nabla_l h_{ij} - \nabla_i h_{jl})(\sqrt{g} - \sqrt{\mathring{g}})) \\ &\quad + \mathring{\nabla}_k\Big((g^{ik} - \mathring{g}^{ik})\hat{\xi}^0_\infty g^{jl}(\nabla_l h_{ij} - \nabla_i h_{jl}) \\ &\quad + \mathring{g}^{ik}\hat{\xi}^0_\infty g^{jl}(\nabla_l h_{ij} - \nabla_i h_{jl})\Big)\sqrt{\mathring{g}}. \end{aligned}$$

We now consider the final term in the above expression, which is also the dominant term:

$$\begin{aligned} \mathring{\nabla}_k\big(\mathring{g}^{ik}\hat{\xi}^0_\infty g^{jl}(\nabla_l h_{ij} - \nabla_i h_{jl})\big) &= \mathring{\nabla}_k\big(\mathring{g}^{ik}\hat{\xi}^0_\infty(g^{jl} - \mathring{g}^{jl})(\nabla_l h_{ij} - \nabla_i h_{jl}) \\ &\quad + \mathring{g}^{ik}\hat{\xi}^0_\infty\mathring{g}^{jl}(\nabla_l h_{ij} - \nabla_i h_{jl})\big) \\ \mathring{\nabla}_k\big(\mathring{g}^{ik}\hat{\xi}^0_\infty\mathring{g}^{jl}(\nabla_l h_{ij} - \nabla_i h_{jl})\big) &= \mathring{\nabla}_k\big(\mathring{g}^{ik}\mathring{g}^{jl}\hat{\xi}^0_\infty((\nabla_l - \mathring{\nabla}_l)h_{ij} - (\nabla_i - \mathring{\nabla}_i)h_{jl}) \\ &\quad \mathring{g}^{ik}\mathring{g}^{jl}\hat{\xi}^0_\infty(\mathring{\nabla}_l h_{ij} - \mathring{\nabla}_i h_{jl})\big). \end{aligned}$$

Note that the final term in the above expression agrees with the first term in (6.18) and therefore cancels. Assembling the pieces above allows us to write the first two terms of (6.18) as follows:

$$-\oint_{S_\infty}\Big(g^{ik}\hat{\xi}^0_\infty g^{jl}(\nabla_l h_{ij}-\nabla_i h_{jl})(\sqrt{g}-\sqrt{\mathring{g}})+(g^{ik}-\mathring{g}^{ik})\hat{\xi}^0_\infty g^{jl}(\nabla_l h_{ij}-\nabla_i h_{jl})\sqrt{\mathring{g}}$$
$$+\mathring{g}^{ik}\mathring{g}^{jl}\hat{\xi}^0_\infty((\nabla_l-\mathring{\nabla}_l)h_{ij}-(\nabla_i-\mathring{\nabla}_i)h_{jl})\sqrt{\mathring{g}}\Big)dS_k. \tag{6.20}$$

Recalling that $g=\mathring{g}+o(r^{-1/2})$ is continuous, and employing Proposition 3.8 we bound the terms in (6.20) as follows:

$$\begin{aligned}
&\oint_{S_R} g^{ik}\hat{\xi}^0_\infty g^{jl}(\nabla_l h_{ij}-\nabla_i h_{jl})(\sqrt{g}-\sqrt{\mathring{g}})dS_k\\
&\qquad\le c\|\hat{\xi}_\infty\|_{\infty:S_R}\|\nabla h\|_{1:S_R}\|\sqrt{g}-\sqrt{\mathring{g}}\|_{\infty:S_R}\\
&\qquad\le \|\hat{\xi}_\infty\|_{\infty:S_R}(\|\mathring{\nabla}h\|_{1:S_R}+\|\tilde{\Gamma}h\|_{1:S_R})\|\sqrt{g}-\sqrt{\mathring{g}}\|_{\infty:S_R}\\
&\qquad= R^{1/2}(\|h\|_{2,2,-1/2}+\|h\|_{\infty:S_R}\|g-\mathring{g}\|_{2,2,-1/2})o(R^{-1/2})\\
&\qquad= o(1),\\
&\oint_{S_R}(g^{ik}-\mathring{g}^{ik})\hat{\xi}^0_\infty g^{jl}(\nabla_l h_{ij}-\nabla_i h_{jl})\sqrt{\mathring{g}}dS_k\\
&\qquad\le c\|\hat{\xi}_\infty\|_{\infty:S_R}\|\nabla h\|_{1:S_R}\|g-\mathring{g}\|_{\infty:S_R}\\
&\qquad= R^{1/2}(\|h\|_{2,2,-1/2}+\|h\|_{\infty:S_R}\|g-\mathring{g}\|_{2,2,-1/2})o(R^{-1/2})\\
&\qquad= o(1),\\
&\oint_{S_R}\mathring{g}^{ik}\mathring{g}^{jl}\hat{\xi}^0_\infty((\nabla_l-\mathring{\nabla}_l)h_{ij}-(\nabla_i-\mathring{\nabla}_i)h_{jl})\sqrt{\mathring{g}}dS_k\\
&\qquad\le c\|\hat{\xi}_\infty\|_{\infty:S_R}\|\tilde{\Gamma}\|_{1:S_R}\|h\|_{\infty:S_R}\\
&\qquad= o(1).
\end{aligned}$$

Since the integrand in (6.20) is clearly $W^{1,2}_{loc}$, Lemma 5.5 ensures that the surface integral at infinity vanishes. It then follows that (6.18) reduces to

$$-\int_{\mathcal{M}}(h,b,p,f)\cdot D\Phi_0^*[\hat{\xi}^0_\infty], \tag{6.21}$$

where $D\Phi_0^*$ is the formal adjoint of $D\Phi_0$; not to be confused with $D\Phi_g^*$.

Similarly, the variation of the fourth and fifth terms in (6.15) are

$$\begin{aligned}\int_{\mathcal{M}} \Big\{ & 2\mathring{\nabla}_i(\hat{\xi}^j_\infty p^i_j) + 2\mathring{\nabla}_j(\hat{\xi}^i_\infty \pi^{jk} h_{ki}) - 2\nabla_i(\hat{\xi}^j_\infty p^i_j) - 2\nabla_i(\pi^{ki} h_{jk}\hat{\xi}^j_\infty) \\ & - \nabla_i(\varepsilon^i_a b^a_j \hat{\xi}^j_\infty) + \nabla_i(\hat{\xi}^i_\infty \varepsilon^j_a b^a_j) - \nabla_i(f^i_a \hat{\xi}^j_\infty A^a_j) \\ & - (h, b, p, f) \cdot D\Phi^*_i[\hat{\xi}^i_\infty] \Big\},\end{aligned} \tag{6.22}$$

where all but the first two terms come from (6.12).

Since π, p, ε and f are densities, the divergences above in (6.22) are independent of the connection used. It follows that the first four terms cancel exactly. The remaining divergences are identical to the form of $\mathcal{B}_2$ considered earlier, and therefore vanish by the same argument. The variation of the fourth and fifth terms in (6.15) then reduce to

$$-\int_{\mathcal{M}} (h, b, p, f) \cdot D\Phi_i(g, A, \pi, \varepsilon)^*[\hat{\xi}^i_\infty]. \tag{6.23}$$

Finally, the variation of the sixth and seventh terms in (6.15) are given by

$$\int_{\mathcal{M}} -\mathring{\nabla}_i(\hat{\xi}^a_\infty f^i_a) + \nabla_i(\hat{\xi}^a_\infty f^i_a) - (h, b, p, f) \cdot D\Phi^*_a[\hat{\xi}^a_\infty]. \tag{6.24}$$

Since f is a density, the divergences again do not depend on the connection and therefore the first two terms in (6.24) cancel exactly, leaving

$$-\int_{\mathcal{M}} (h, b, p, f) \cdot D\Phi^*_a[\hat{\xi}^a_\infty]. \tag{6.25}$$

Combining (6.21), (6.23) and (6.25) establishes (6.16). □

Heuristically, the modified Hamiltonian is a Lagrange function; critical points of the Hamiltonian should correspond to critical points of $\xi_\infty \cdot \mathbb{P}$, subject to the constraints being satisfied. In light of Theorem 6.3, it is expected that these critical points correspond to initial data, $(g, A, \pi, \varepsilon) \in \mathcal{F}$, with nontrivial solutions to $D\Phi^*[\xi] = 0$. While corollary 4.12 states that $D\Phi^*$ has trivial kernel, this is only when $D\Phi^*$ is considered as an operator on $L^2_{-1/2}$. If the domain of $D\Phi^*$ is extended to allow ξ to be nonzero at infinity, then the kernel is no longer trivial; recall, if $D\Phi^*_{(g,A,\pi,\varepsilon)}[\xi] = 0$ for some $\xi \neq 0$ then (g, A, π, ε) is a stationary initial data set. Furthermore, if ξ^μ_∞ is parallel to $\mathbb{P}^\mu$, unit speed and future-pointing, then $\xi_\infty \cdot \mathbb{P} = m + V_\infty \cdot Q_\infty$. This motivates Theorem 6.6 below, which is established by rigorously constructing the aforementioned Lagrange multipliers argument.

For this, we need the following generalisation of the method of Lagrange multipliers to Banach manifolds (see Theorem 6.3 of [11]).

Theorem 6.4. *Suppose $K : B_1 \to B_2$ is a C^1 map between Banach manifolds, such that $DK_u : T_uB_1 \to T_{K(u)}B_2$ is surjective, with closed kernel and closed complementary subspace for all $u \in K^{-1}(0)$. Let $f \in C^1(B_1)$ and fix $u \in K^{-1}(0)$, then the following statements are equivalent:*

(i) For all $v \in \ker DK_u$, we have

$$Df_u(v) = 0 \tag{6.26}$$

(ii) There is $\lambda \in T^*_{K(u)}B_2$ such that for all $v \in T_uB_1$,

$$Df_u(v) = \langle \lambda, DK_u(v) \rangle \,, \tag{6.27}$$

where $\langle \, , \rangle$ refers to the natural dual pairing.

The proof of Theorem 6.6 will also require the following lemma, showing that if $\xi \in W^{2,2}_{\xi_\infty}$, then $f = D\Phi^*[\xi]$ satisfies the hypotheses of Theorem 4.10.

Lemma 6.5. *For $(g, A, \pi, \varepsilon) \in \mathcal{F}$ and $\xi \in W^{2,2}_{\xi_\infty}$, we have*

$$D\Phi^*_g[\xi] \in L^2_{-5/2}(S^2 \otimes \Lambda^3(\mathcal{M})), \qquad D\Phi^*_A[\xi] \in L^2_{-3/2}(T\mathcal{M} \otimes \mathfrak{g} \otimes \Lambda^3(\mathcal{M})),$$
$$D\Phi^*_\pi[\xi] \in W^{1,2}_{-3/2}(S_2), \qquad D\Phi^*_\varepsilon[\xi] \in W^{1,2}_{-3/2}(T^*\mathcal{M} \otimes \mathfrak{g}).$$

Proof. Recall that $\hat{\xi}_\infty$ is smooth, $\mathring{\nabla}\hat{\xi}_\infty$ has compact support and $\|\hat{\xi}_\infty\|_{\infty,0}$ is finite. Since $\xi - \hat{\xi}_\infty \in W^{2,2}_{-1/2}$, it follows that we have $\mathring{\nabla}\xi \in W^{1,2}_{-3/2}$, and the usual weighted inequalities give the following estimates:

$$\begin{aligned}
\|D\Phi^*_g[\xi]\|_{2,-5/2} \leq c\Big(&\|N\|_{\infty,0}(\|\pi^2\|_{2,-5/2} + \|E^2\|_{2,-5/2} + \|B^2\|_{2,-5/2} \\
&+ \|Ric\|_{2,-5/2}) + \|X\|_{\infty,0}\|\mathring{\nabla}\pi\|_{2,-5/2} + \|\mathring{\nabla}^2 N\|_{2,-5/2} \\
&+ \|X\|_{4,-3/2}\|\varepsilon\|_{4,-3/2}\Big) \\
\leq c\Big(&\|\xi\|_{\infty,0}\big((1 + \|\pi\|_{1,2,-3/2})^2 + \|E\|^2_{1,2,-3/2} + \|B\|^2_{1,2,-3/2} \\
&+ \|Ric\|_{2,-5/2}\big) + \|\mathring{\nabla}^2 N\|_{2,-5/2} + \|X\|_{1,2,-3/2}\|\varepsilon\|_{1,2,-3/2}\Big),
\end{aligned}$$

$$\begin{aligned}
\|D\Phi^*_A[\xi]\|_{2,-3/2} \leq c\Big(&\|N\|_{\infty,0}(\|\mathring{\nabla}B\|_{2,-3/2} + \|\tilde{\Gamma}B\|_{2,-3/2} + \|BA_{\mathfrak{k}}\|_{2,-3/2}) \\
&+ \|X\|_{\infty,0}\|\mathring{\nabla}\varepsilon\|_{2,-3/2} + \|\mathring{\nabla}N\|_{4,0}\|B\|_{4,-3/2} \\
&+ \|\mathring{\nabla}X\|_{4,0}\|\varepsilon\|_{4,-3/2} + \|V\|_{\infty,0}\|\varepsilon\|_{2,-3/2}\Big) \\
\leq c\Big(&\|\xi\|_{\infty,0}\big((1 + \|\mathring{\nabla}g\|_{4,0} + \|A_{\mathfrak{k}}\|_{4,0})\|B\|_{1,2,-3/2} \\
&+ \|\varepsilon\|_{1,2,-3/2}\big) + \|\mathring{\nabla}\xi\|_{1,2,0}(\|\varepsilon\|_{1,2,-3/2} + \|B\|_{1,2,-3/2})\Big) \\
\leq c\Big(&\|\xi\|_{\infty,0}\big((1 + \|\mathring{\nabla}g\|_{1,2,-3/2} + \|A_{\mathfrak{k}}\|_{1,2,-1/2})\|B\|_{1,2,-3/2} \\
&+ \|\varepsilon\|_{1,2,-3/2}\big) + \|\mathring{\nabla}\xi\|_{1,2,0}(\|\varepsilon\|_{1,2,-3/2} + \|B\|_{1,2,-3/2})\Big),
\end{aligned}$$

$$\begin{aligned}
\|\mathring{\nabla}(D\Phi^*_\pi[\xi])\|_{2,-5/2} \leq c\Big(&\|N\|_{\infty,0}\big(\|\mathring{\nabla}g\|_{4,-3/2}\|\pi\|_{4,-1} + \|\mathring{\nabla}\pi\|_{2,-5/2}\big) \\
&+ \|\mathring{\nabla}N\|_{4,-1}\|\pi\|_{4,-3/2} + \|\mathring{\nabla}^2 X\|_{2,-5/2} \\
&+ \|\mathring{\nabla}X\|_{4,-1}\|\mathring{\nabla}g\|_{4,-3/2} + \|X\|_{\infty,0}\|\mathring{\nabla}^2 g\|_{2,-5/2}\Big) \\
\leq c\Big(&\|\xi\|_{\infty,0}\big(\|\mathring{\nabla}g\|_{1,2,-3/2}\|\pi\|_{1,2,-3/2} + \|\mathring{\nabla}\pi\|_{2,-5/2} \\
&+ \|\mathring{\nabla}^2 g\|_{2,-5/2}\big) + \|\mathring{\nabla}^2 X\|_{2,-5/2} \\
&+ \|\mathring{\nabla}\xi\|_{1,2,-1}\big(\|\pi\|_{1,2,-3/2} + \|\mathring{\nabla}g\|_{1,2,-3/2}\big)\Big).
\end{aligned}$$

Note that we must keep track of raising and lowering indices above, as derivatives of g are important. We have also made use of the identity $\delta\det(g) = \det(g)g^{ij}\delta g_{ij}$. Finally, we have

$$\begin{aligned}
\|\mathring{\nabla}(D\Phi^*_\varepsilon[\xi])\|_{2,-5/2} \leq c\Big(& \|N\|_{\infty,0}\big(\|\mathring{\nabla} g\|_{4,-3/2}\|\varepsilon\|_{4,-1} + \|\mathring{\nabla}\varepsilon\|_{2,-5/2}\big) \\
&+ \|\mathring{\nabla} N\|_{4,-1}\|\varepsilon\|_{4,-3/2} + \|X\|_{\infty,0}\|\mathring{\nabla}^2 A\|_{2,-5/2} \\
&+ \|\mathring{\nabla} X\|_{4,-1}\|\mathring{\nabla} A\|_{4,-3/2} + \|\mathring{\nabla}^2 X\|_{2,-2}\|A\|_{\infty,-1/2} \\
&+ \|V\|_{\infty,0}\|\mathring{\nabla}^2 A_{\mathfrak{k}}\|_{2,-5/2} + \|\mathring{\nabla} V\|_{4,0}\|\mathring{\nabla} A_{\mathfrak{k}}\|_{4,-5/2} \\
&+ \|\mathring{\nabla}^2 V\|_{2,-1}\|A_{\mathfrak{k}}\|_{\infty,-3/2}\Big) \\
\leq c\Big(& \|\xi\|_{\infty,0}\big(\|\mathring{\nabla} g\|_{4,-3/2}\|\varepsilon\|_{4,-1} + \|\mathring{\nabla}\varepsilon\|_{2,-5/2} \\
&+ \|\mathring{\nabla}^2 A\|_{2,-5/2}\big) + \|\mathring{\nabla}\hat{\xi}_\infty\|_{1,2,-1}\big(\|\varepsilon\|_{1,2,-3/2} \\
&+ \|\mathring{\nabla} A\|_{1,2,-3/2}\big) + \|\mathring{\nabla}^2\xi\|_{2,-2}\|A\|_{2,2,-1/2}\Big),
\end{aligned}$$

and applying the weighted Poincaré inequality completes the proof. □

The main result, equating the validity of the first law with stationary solutions, is a corollary of the following Theorem. We will make use of the shorthand, $G = (g, A, \pi, \varepsilon) \in \mathcal{F}$, to indicate the point that we are linearising about, as this will be important in proof below.

Theorem 6.6. *Take $G \in \mathcal{F}$ such that $\Phi(G) = s \in L^1$. Let $\xi_\infty \in \mathbb{R}^{3,1} \oplus \mathfrak{g}$ be fixed and define the energy functional $E \in C^\infty(\mathcal{C}(c))$ by*

$$E^{\xi_\infty}(G) = \xi_\infty \cdot \mathbb{P}(G). \tag{6.28}$$

The following statements are equivalent:

(i) For all $(h, b, p, f) \in T_G\mathcal{C}(s)$,

$$DE^{\xi_\infty}_G[h, b, p, f] = 0. \tag{6.29}$$

(ii) There is $\xi \in W^{2,2}_{\xi_\infty}(\Lambda^0(\mathcal{M}) \times T\mathcal{M} \times \mathfrak{g})$ *satisfying*

$$D\Phi^*_G[\xi] = 0. \tag{6.30}$$

Proof. First we show $(i) \Rightarrow (ii)$: that is, assume (i) holds for some fixed s and $\tilde{G} = (\tilde{g}, \tilde{A}, \tilde{\pi}, \tilde{\varepsilon}) \in \mathcal{F}$. Fix some $\tilde{\xi} \in W^{2,2}_{\xi_\infty}$, and define $f(G) := \mathcal{H}(G; \tilde{\xi})$ and $K(G) := \Phi(G) - s$; these maps satisfy the hypotheses of Theorem 6.4. Note that $T_G\mathcal{C}(s) = \ker(D\Phi_G) = \ker(DK_G)$ and we have $Df_G = 16\pi DE_G$ on $T_G\mathcal{C}(s)$. It follows that statement (i) of Theorem 6.6 implies statement (i) of Theorem 6.4, and therefore there exists $\lambda \in \mathcal{N}$ such that for all $(h, b, p, f) \in T_{\tilde{G}}\mathcal{F}$,

$$\begin{aligned} Df_{\tilde{G}}[h, b, p, f] &= \langle \lambda, D\Phi_{\tilde{G}}[h, b, p, f] \rangle \\ &= \int_{\mathcal{M}} \lambda \cdot D\Phi_{\tilde{G}}[h, b, p, f]. \end{aligned}$$

Now from Theorem 6.3, we have $Df_{\tilde{G}}[h, b, p, f] = -\int_{\mathcal{M}} (h, b, p, f) \cdot D\Phi^*_{\tilde{G}}[\xi]$. In particular, we have

$$-\int_{\mathcal{M}} (h, b, p, f) \cdot D\Phi^*_{\tilde{G}}[\xi] = \int_{\mathcal{M}} \lambda \cdot D\Phi_{\tilde{G}}[h, b, p, f], \tag{6.31}$$

for all $(h, b, p, f) \in T_{\tilde{G}}\mathcal{F}$. That is, λ satisfies

$$D\Phi^*_{\tilde{G}}[\lambda] = -D\Phi^*_{\tilde{G}}[\tilde{\xi}] \tag{6.32}$$

in the weak sense.

Now by Lemma 6.5, (6.32) satisfies the hypothesis of Theorem 4.10 and therefore $\lambda \in W^{2,2}_{-1/2}(\Lambda^0 \times T\mathcal{M} \times \mathfrak{g} \otimes \Lambda^0)$ is a strong solution to (6.32); i.e., $\xi := \lambda + \tilde{\xi} \in W^{2,2}_{\xi_\infty}$ satisfies

$$D\Phi^*_{\tilde{G}}[\xi] = 0, \tag{6.33}$$

strongly. That is, $\tilde{G}$ is a generalised stationary initial data set with generalised Killing vector, ξ, and we conclude $(i) \Rightarrow (ii)$.

Conversely, assume $D\Phi_G^*[\xi] = 0$ for some $\xi \in W^{2,2}_{\xi_\infty}$ and $G \in \mathcal{F}$. For all $(h,b,p,f) \in T_G\mathcal{C}(s)$, we have

$$16\pi DE_G^{\xi_\infty}(\xi)[h,b,p,f] = D\mathcal{H}_G(\xi)[h,b,p,f].$$

Now from Theorem 6.3 we also have

$$D\mathcal{H}_G(\xi)[h,b,p,f] = -\int_{\mathcal{M}} (h,b,p,f) \cdot D\Phi_G^*[\xi] = 0$$

for all $(h,b,p,f) \in T_G\mathcal{F}$. It follows that

$$16\pi DE_G(\xi)[h,b,p,f] = D\mathcal{H}_G(\xi)[h,b,p,f] = 0$$

for all $(h,b,p,f) \in T_G\mathcal{C}(s)$. □

A result of Beig and Chruściel [13] states that if a Killing vector, ξ^μ, is timelike at infinity, then it is asymptotically proportional to $\mathbb{P}^\mu = \eta^{\mu\nu}\mathbb{P}_\nu$, where η is the Minkowski metric. Recalling that the mass is given by $m = \sqrt{-\mathbb{P}^\mu\mathbb{P}_\mu}$, and rescaling ξ appropriately, we can write $m = \xi^\mu_\infty\mathbb{P}_\mu$. From this, the following Corollary of Theorem 6.6 is obtained.

Corollary 6.7. *Suppose $G \in \mathcal{C}(s)$, $s \in L^1$ and $\mathbb{P}^\mu$ is a past-pointing timelike vector in the spacetime, then the following statements are equivalent:*

(i) For all $(h,b,p,f) \in T_G\mathcal{C}(s)$,

$$Dm_G[h,b,p,f] + V_\infty \cdot DQ_G[h,b,p,f] = 0, \tag{6.34}$$

where m is the mass, Q_a is the Yang-Mills electric charge and $V_\infty \in \mathfrak{g}$.

(ii) G is a generalised stationary initial data set with infinitesimal symmetry, ξ, in the sense $D\Phi_G^*[\xi] = 0$. Furthermore, ξ_∞ is proportional to $(\mathbb{P}^\mu, -mV^a_\infty)$.

Proof. Assume (i) holds. Let $\xi_\infty = -\frac{1}{m}(\mathbb{P}^\mu, -mV^a)$, normalising ξ^μ_∞ to be a future-pointing unit vector. We have $E^{\xi_\infty} = \xi^\alpha \mathbb{P}_\alpha = m + V^a_\infty Q_a$, and condition (i) implies that $DE^{\xi_\infty}_G[h, b, p, f] = 0$ for all $(h, b, p, f) \in T_G\mathcal{C}(s)$. It follows that (ii) is an immediate consequence of Theorem 6.6.

Conversely, assume condition (ii) holds, then by rescaling if necessary, we again have $\xi_\infty = -\frac{1}{m}(\mathbb{P}^\mu, -mV^a)$. From Theorem 6.6 we infer $DE^{\xi_\infty}_G[h, b, p, f] = 0$ for all $(h, b, p, f) \in T_G\mathcal{C}(s)$, and therefore (i) holds. □

6.3 With an Interior Boundary

In this section, we show that the full first law gives a condition for stationarity when the Cauchy surface has a closed 2-surface interior boundary. It may then seem natural, to repeat the analysis of Chapter 4 on a manifold with boundary. However, the analysis of PDEs on domains with boundary is significantly different from the case of no boundary, and much of the analysis runs into difficulties. It may be possible to use weights to control the behaviour of functions as they approach the interior boundary, however this approach also has difficulties. These problems are circumvented entirely by considering the same phase space as in the previous case, but only considering evolution on a single end. Specifically, we will consider evolution on $\mathcal{M}_0$, defined in Chapter 5 (above (5.7)), and let Σ be the interior boundary surface.

In the preceding section, the variation of the mass and charge arose as a boundary term at infinity from the difference, $(h, b, p, f) \cdot D\Phi^*[\xi] - \xi \cdot D\Phi[h, b, p, f]$. As we are concerned with the evolution on $\mathcal{M}_0$, boundary terms are also present on Σ. It is shown that under suitable boundary conditions for ξ, these extra boundary terms correspond to the remaining terms in the first law.

Consider now, the Einstein-Yang-Mills pure constraint Hamiltonian on $\mathcal{M}_0$:

$$\hat{\mathcal{H}}^{\xi}_{EYM}(g, A, \pi, \varepsilon) = -\int_{\mathcal{M}_0} \Phi(g, A, \pi, \varepsilon) \cdot \xi. \tag{6.35}$$

We first note that this Hamiltonian is a smooth function on the phase space.

Proposition 6.8. *The map $\hat{\mathcal{H}}_{EYM} : \mathcal{F} \times \mathcal{N}^* \to \mathbb{R}$ is smooth in the sense of Fréchet differentiability.*

Proof. This follows directly from the proof of Proposition 6.1 without modification. $\square$

The following proposition, analogous to Theorem 6.2, demonstrates that the pure constraint Hamiltonian gives the correct equations of motion, when ξ vanishes at infinity and on the boundary. Recall, we use $\mathring{W}^{k,p}_{\delta}$ to denote the completion of C^{∞}_{c} with respect to the $W^{k,p}_{\delta}$ norm.

Proposition 6.9. *For $\xi \in \mathring{W}^{2,2}_{-1/2}\big(\Lambda^0(\mathcal{M}_0) \times T\mathcal{M}_0 \times \mathfrak{g} \otimes \Lambda^0(\mathcal{M}_0)\big)$, we have*

$$D\hat{\mathcal{H}}^{\xi}_{EYM} \cdot (h, b, p, f) = -\int_{\mathcal{M}_0} D\Phi^*[\xi] \cdot (h, b, p, f),$$

for all $(h, b, p, f) \in T_{(g,A,\pi,\varepsilon)}\mathcal{F}$.

Proof. This also follows easily from the previous case. We must show

$$\int_{\mathcal{M}_0} \xi \cdot D\Phi[h, b, p, f] = \int_{\mathcal{M}_0} (h, b, p, f) \cdot D\Phi^*[\xi],$$

which amounts to proving that the boundary terms arising from the integral of equation (6.12) vanish. The boundary terms at infinity are exactly those considered in the previous case so these terms vanish by Theorem 6.2. By hypothesis, ξ and $\mathring{\nabla}\xi$ vanish on Σ in the trace sense so the boundary terms arising on Σ also vanish. $\square$

Proposition 6.9 is equivalent to the statement that this Hamiltonian gives the correct equations of motion. However, since ξ represents the evolution vector, Proposition 6.9 pertains only to the evolution of data that is fixed at infinity and on Σ. Without these boundary conditions, the cumbersome boundary terms from (6.12) do not vanish in general. However, if (g, A, π, ε) is a stationary, axially symmetric black hole initial data set that intersects the bifurcation surface of a bifurcate Killing horizon, then these boundary terms have an interesting geometric interpretation.

Let ξ^μ be the stationary Killing vector which is future pointing and unit-timelike at infinity, and let ϕ be the axial Killing vector with 2π-periodic orbits, tangent to the Cauchy surface. Further, suppose that Σ is the bifurcation surface of the black hole. It is well known that there exists a constant, Ω, such that $\xi^\mu + \Omega\phi = 0$ on Σ; the angular velocity of the black hole. The zeroth law states that the surface gravity κ is constant on the horizon, and in some sense there is an analogous Yang-Mills zeroth law. That is, the electric potential is constant on the horizon under appropriate gauge conditions [7].

The boundary terms at infinity from (6.12) are the same as those in the preceding section, so we need only examine the terms arising on Σ. Since ϕ is tangential to the embedding of $\mathcal{M}$ in the full spacetime, the condition on Σ becomes $N = 0$ and $X = -\Omega\phi$. In particular, X is tangential to Σ, so the boundary terms reduce to

$$\begin{aligned} -\oint_\Sigma \big((\mathring{\nabla}^j(N)h_{ij} - \mathrm{tr}_g\, h\mathring{\nabla}_i(N))\sqrt{g} - 2X^j p_{ij} + V^a f_{ai} \\ - 2\pi_i^k h_{jk} X^j + \varepsilon_{ia} b_j^a X^j + f_{ia} X^j A_j^a\big) n^i. \end{aligned} \tag{6.36}$$

The negative sign outside the integral serves as a reminder that the normal vector we use is the negative of what appears in the divergence theorem, when applied to (6.12).

The surface gravity is given by $\kappa = n^i \mathring{\nabla}_i(N)$, which is evaluated on Σ [57], and since it is constant on a bifurcate Killing horizon, $\mathring{\nabla} N$ is normal to Σ. Making use of co-ordinates adapted to Σ, the first two terms in (6.36) can be interpreted as follows:

$$
\begin{aligned}
-\oint_\Sigma &\left(\mathring{\nabla}^j(N)h_{ij} - \mathrm{tr}_g h \mathring{\nabla}_i(N)\right)\sqrt{g}dS^i \\
&= -\oint_\Sigma \left(g^{j3}\mathring{\nabla}_3(N)h_{ij}n^i - h^k_k \mathring{\nabla}_3(N)\right)\sqrt{g}dS \\
&= \oint_\Sigma \mathring{\nabla}_3(N)h^A_A \sqrt{g}dS \\
&= 2\kappa dA, \qquad (6.37)
\end{aligned}
$$

where the index 3 refers to the direction normal to Σ and $A = 1, 2$ indicates tangential indices.

The Yang-Mills contribution to the first law comes from the expression,

$$
-\oint_\Sigma V^a f_{ai} n^i = -V^a_\Sigma \oint_\Sigma f_{ai} n^i = 16\pi V_\Sigma \cdot dQ_\Sigma,
$$

where we have assumed the gauge condition ensuring $V \equiv V_\Sigma \in \mathfrak{g}$ is constant on the horizon. The remaining terms in (6.36) are proportional to the variation of the angular momentum:

$$
dJ = d\tilde{J}_\phi = -\frac{1}{16\pi}\oint_\Sigma (2\phi^j p_{ij} + 2\pi^k_i h_{jk}\phi^j - \varepsilon_{ia} b^a_j \phi^j - f_{ia}\phi^j A^a_j)n^i.
$$

Since that on the horizon we have $X = -\Omega\phi$, it is clear that these remaining terms equal $-16\pi\Omega$ times the variation of angular momentum of Σ.

Recalling that the boundary terms at infinity give the variation of the mass and the charge at infinity, we find that the combined boundary terms from (6.12) equal

$$
16\pi(dm - \frac{\kappa}{8\pi}dA - \Omega dJ - V_\Sigma \cdot dQ_\Sigma + V_\infty \cdot dQ_\infty). \qquad (6.38)
$$

This suggests that the differential relationship given by the first law (2.18) must hold, in order for the pure constraint Hamiltonian to give the correct equations of motion. This has been rigorously established by Ashtekar, Fairhurst and Krishnan, where the evolution is considered exterior to an isolated horizon [7]. However, we take a different approach to these boundary terms and the first law plays quite a different role here. As in the previous section, we redefine the Hamiltonian à la Regge-Teitelboim by adding boundary terms, making the equations of motion valid more generally.

Fix $\hat{\xi}_\infty$, $\hat{\xi}_\Sigma$ and $\hat{\xi}_{\mathrm{ref}}$, as in the preceding Chapter, with $\hat{\xi}^i_\Sigma$ tangent to Σ, $\hat{\xi}^a_\Sigma = \xi^a_\Sigma \in \mathfrak{g}$ constant, and $\hat{\xi}^0_\Sigma = 0$. These boundary conditions will be assumed throughout the remainder of this thesis, unless stated otherwise. Recall the spaces, $L^p_{\hat{\xi}_{\mathrm{ref}}}$ (5.20), $W^{k,p}_{\hat{\xi}_{\mathrm{ref}}}$ (5.21), which we use to enforce the boundary conditions and asymptotics. For $\xi \in L^2_{\hat{\xi}_{\mathrm{ref}}}$ we define the modified Hamiltonian,

$$\hat{\mathcal{H}}^\xi_{RT}(g, A, \pi, \varepsilon) := 16\pi(\xi_\infty \cdot \mathbb{P}' + \tilde{J}_\xi - \xi^a_\Sigma Q_{\Sigma\, a}) - \int_{\mathcal{M}_0} \xi \cdot \Phi(g, A, \pi, \varepsilon), \tag{6.39}$$

where $\mathbb{P}'$ is defined by equations (5.22)-(5.24). As in the previous section, this Hamiltonian is not well-defined on the entire phase space; both integrals diverge in general. However, as the surface integrals at infinity are identical to those considered in the previous section, the dominant terms at infinity still cancel out and a regularised Hamiltonian is defined everywhere on $\mathcal{F}$. By construction, the regularised Hamiltonian has almost identical form to (6.15):

$$\begin{aligned}\hat{\mathcal{H}}^\xi(g, A, \pi, \varepsilon) = &\int_{\mathcal{M}_0} (\hat{\xi}_{\mathrm{ref}} - \xi) \cdot \Phi + \int_{\mathcal{M}_0} \hat{\xi}^0_{\mathrm{ref}} \mathring{g}^{ik} \mathring{g}^{jl} (\mathring{\nabla}_k \mathring{\nabla}_l g_{ij} - \mathring{\nabla}_i \mathring{\nabla}_k (g_{jl}) \sqrt{\mathring{g}} - \Phi_0) \\ &+ \int_{\mathcal{M}_0} \mathring{g}^{ik} \mathring{g}^{jl} \mathring{\nabla}_k \hat{\xi}^0_{\mathrm{ref}} (\mathring{\nabla}_l g_{ij} - \mathring{\nabla}_i g_{jl}) \sqrt{\mathring{g}} \\ &+ \int_{\mathcal{M}_0} \hat{\xi}^i_{\mathrm{ref}} (2\mathring{\nabla}_j \pi^j_i + \mathring{\nabla}_j (\varepsilon^j_a A^a_i) - \Phi_i) + \int_{\mathcal{M}_0} (2\pi^j_i + \varepsilon^j_a A^a_i) \mathring{\nabla}_j \hat{\xi}^i_{\mathrm{ref}} \\ &+ \int_{\mathcal{M}_0} \hat{\xi}^a_{\mathrm{ref}} (4\mathring{\nabla}_i E^i_a - \Phi_a) + 4 \int_{\mathcal{M}_0} E^i_a \mathring{\nabla}_i \hat{\xi}^a_{\mathrm{ref}},\end{aligned} \tag{6.40}$$

which is again defined for each $\xi \in L^2_{\hat{\xi}_{\mathrm{ref}}}$. The additional terms appearing in this Hamiltonian are momentum terms for the Yang-Mills fields. As they do not contribute anything at infinity, they were not included in the preceding case (recall the discussion under the definition of $\mathbb{P}'_i$ (5.23)).

Proposition 6.10. *The regularised Hamiltonian,* $\hat{\mathcal{H}} : \mathcal{F} \times L^2_{\hat{\xi}_{\mathrm{ref}}} \to \mathbb{R}$, *is well-defined and smooth.*

Proof. This is almost identical to the case where the Hamiltonian is defined over $\mathcal{M}$, rather than $\mathcal{M}_0$. The estimates used in Theorem 6.3, to show that the previous regularised Hamiltonian was well-defined and smooth, are still valid on $\mathcal{M}_0$ as the weighted Sobolev and Hölder inequalities are valid on a manifold with boundary. All that remains to be established, is that the additional terms, $\int_{\mathcal{M}_0} \hat{\xi}^i_{\mathrm{ref}} \mathring{\nabla}_j(\varepsilon^j_a A^a_i)$ and $\int_{\mathcal{M}_0} \varepsilon^j_a A^a_i \mathring{\nabla}_j \hat{\xi}^i_{\mathrm{ref}}$, are locally bounded, and then we conclude that $\hat{\mathcal{H}}$ is smooth. The latter term is bounded since $\mathring{\nabla}\hat{\xi}_{\mathrm{ref}}$ is compactly supported, and for the former we have

$$\|\hat{\xi}_{\mathrm{ref}} \mathring{\nabla}(\varepsilon A)\|_{1,-3} \leq c\|\hat{\xi}_{\mathrm{ref}}\|_{\infty,0}(\|\mathring{\nabla}\varepsilon\|_{2,-5/2}\|A\|_{2,-1/2} + \|\varepsilon\|_{2,-3/2}\|\mathring{\nabla}A\|_{2,-3/2}).$$

$\square$

The new Hamiltonian contains all of the quantities arising in the first law except for the area term (see 6.39), which is not included since κdA is not the first variation of any quantity; κ depends explicitly on g, and therefore $d(\kappa A) \neq \kappa dA$. This is an obstruction to generating the correct equations of motion from our new Hamiltonian, which is evident in the following Proposition, analogous to Theorem 6.3.

Proposition 6.11. *For $\xi \in W^{2,2}_{\hat{\xi}_{\mathrm{ref}}}$, the variation of the regularised Hamiltonian is given by*

$$D\hat{\mathcal{H}}^{\xi}[h,b,p,f] = -\oint_{\Sigma} (\mathring{\nabla}^j(\xi^0)h_{ij} - tr_g h\mathring{\nabla}_i(\xi^0))\sqrt{g}dS^i \\ - \int_{\mathcal{M}_0} D\Phi^*[\xi]\cdot(h,b,p,f). \tag{6.41}$$

Proof. As in the proof of Theorem 6.3, we consider the terms in (6.40) separately. By Proposition 6.9, the variation of the first integral in (6.40) becomes

$$\int_{\mathcal{M}_0} (h,b,p,f)\cdot D\Phi^*[\hat{\xi}_{\mathrm{ref}} - \xi].$$

The variation of the second and third terms combine (cf. (6.18)) to give

$$\int_{\mathcal{M}_0} \Big\{\mathring{g}^{ik}\mathring{\nabla}_k(\hat{\xi}^0_{\mathrm{ref}}\mathring{g}^{jl}(\mathring{\nabla}_l h_{ij} - \mathring{\nabla}_i h_{jl}))\sqrt{\mathring{g}} - \nabla^i(\hat{\xi}^0_{\mathrm{ref}}(\nabla^j h_{ij} - \nabla_i \operatorname{tr}_g h))\sqrt{g} \\ + \nabla^i(h_{ij}\nabla^j\hat{\xi}^0_{\mathrm{ref}} - \operatorname{tr}_g h\nabla_i\hat{\xi}^0_{\mathrm{ref}})\sqrt{g} - (h,b,p,f)\cdot D\Phi^*_0[\hat{\xi}^0_{\mathrm{ref}}]\Big\}. \tag{6.42}$$

Then the first two terms in the above combine to give a total divergence (cf. (6.20)),

$$-\oint_{\mathcal{M}_0} \mathring{\nabla}_k\Big(g^{ik}\hat{\xi}^0_{\mathrm{ref}}g^{jl}(\nabla_l h_{ij} - \nabla_i h_{jl})(\sqrt{g} - \sqrt{\mathring{g}}) + (g^{ik} - \mathring{g}^{ik})\hat{\xi}^0_{\mathrm{ref}}g^{jl}(\nabla_l h_{ij} - \nabla_i h_{jl})\sqrt{\mathring{g}} \\ + \mathring{g}^{ik}\mathring{g}^{jl}\hat{\xi}^0_{\mathrm{ref}}((\nabla_l - \mathring{\nabla}_l)h_{ij} - (\nabla_i - \mathring{\nabla}_i)h_{jl})\sqrt{\mathring{g}}\Big), \tag{6.43}$$

which is rewritten as surface integrals, both at infinity and on Σ. The integral at infinity is identical to that considered in the proof of Theorem 6.3 and therefore vanishes by the same argument, while the surface integral on Σ vanishes since $\xi^0_{\Sigma} = 0$. The third term in (6.42) is again a divergence, but only gives a boundary term on Σ since $\mathring{\nabla}\hat{\xi}_{\mathrm{ref}}$ has bounded support. This boundary term on Σ is then exactly the surface integral in (6.41).

The variation of the fourth and fifth terms in (6.40) give

$$\begin{aligned}\int_{\mathcal{M}_0} \Big\{ & 2\mathring{\nabla}_i(\hat{\xi}^j_{\text{ref}} p^i_j) + 2\mathring{\nabla}_j(\hat{\xi}^i_{\text{ref}} \pi^{jk} h_{ki}) + \nabla_i(\varepsilon^i_a b^a_j \hat{\xi}^j_{\text{ref}}) + \nabla_i(f^i_a \hat{\xi}^j_{\text{ref}} A^a_j) \\ & - 2\nabla_i(\hat{\xi}^j_{\text{ref}} p^i_j) - 2\nabla_i(\pi^{ki} h_{jk} \hat{\xi}^j_{\text{ref}}) - \nabla_i(\varepsilon^i_a b^a_j \hat{\xi}^j_{\text{ref}}) - \nabla_i(f^i_a \hat{\xi}^j_{\text{ref}} A^a_j) \\ & + \nabla_i(\hat{\xi}^i_{\text{ref}} \varepsilon^j_a b^a_j) - (h, b, p, f) \cdot D\Phi^*_i[\hat{\xi}^i_{\text{ref}}] \Big\},\end{aligned} \tag{6.44}$$

which only differs from (6.22) by Yang-Mills angular momentum terms. Since p, π, f and ε are densities, the divergences above do not depend on the connection used and thus the first two lines in (6.44) cancel exactly. The remaining divergence is exactly of the form of $\mathcal{B}_2$, used in the proof of Theorem 6.2, and therefore the surface integral at infinity vanishes by the same argument. Since $\hat{\xi}^i_{\text{ref}}$ is tangent to Σ, the surface integral on Σ also vanishes, and the variation of the fourth and fifth terms in (6.40) become

$$-\int_{\mathcal{M}_0} (h, b, p, f) \cdot D\Phi^*_i[\hat{\xi}^i_{\text{ref}}]. \tag{6.45}$$

The final two terms in (6.40) are exactly the same as those considered in Theorem 6.3 and therefore cancel identically, leaving

$$-\int_{\mathcal{M}_0} (h, b, p, f) \cdot D\Phi^*_a[\hat{\xi}^a_{\text{ref}}]. \tag{6.46}$$

Assembling all of the pieces completes the proof. □

It is interesting to note here, if both ξ^0 and $\mathring{\nabla}\xi^0$ are set to zero on on Σ, then the regularised Hamiltonian does indeed give the correct equations of motion. In particular, if we consider evolution vectors, ξ, that are supported away from Σ then the evolution is Hamiltonian. However, the fact that the evolution is not Hamiltonian is tangential to this thesis; we simply use the Hamiltonian as a Lagrange function.

We are now in a position to prove the main result of this section:

Theorem 6.12. *Let $(g, A, \pi, \varepsilon) \in \mathcal{C}(s)$, where $s \in L^1$, and suppose there exists a vector field, $\phi \in W^{2,2}_{loc}$, tangent to Σ with $D\Phi^*_\pi[\phi], D\Phi^*_\varepsilon[\phi] \in \mathring{W}^{1,2}_{-1/2}(\mathcal{M}_0)$. Further suppose that for all $(h, b, p, f) \in T_{(g,A,\pi,\varepsilon)}\mathcal{C}(s)$,*

$$\begin{aligned} Dm_{(g,A,\pi,\varepsilon)}[h, b, p, f] &= \alpha D\,\mathrm{Area}_{(g,A,\pi,\varepsilon)}[h, b, p, f] + \beta DJ_{\phi\,(g,A,\pi,\varepsilon)}[h, b, p, f] \\ &\quad + \gamma_\Sigma \cdot DQ_{\Sigma\,(g,A,\pi,\varepsilon)}[h, b, p, f] - \gamma_\infty \cdot DQ_{\infty\,(g,A,\pi,\varepsilon)}[h, b, p, f], \end{aligned} \tag{6.47}$$

where $\alpha, \beta \in \mathbb{R}$ and $\gamma_\Sigma, \gamma_\infty \in \mathfrak{g}$ are constants. Then (g, A, π, ε) is a generalised stationary initial data set. Furthermore, γ is the electric potential, and if Σ is the bifurcation surface of a bifurcate Killing horizon, then $8\pi\alpha$ is the surface gravity and β is the angular velocity.

Proof. Assume (6.47) holds at some fixed point $\tilde{G} = (\tilde{g}, \tilde{A}, \tilde{\pi}, \tilde{\varepsilon}) \in \mathcal{F}$. Then fix $\hat{\xi}_{\mathrm{ref}}$ such that it satisfies the following boundary conditions:

- ξ^μ_∞ corresponds to a future pointing unit vector at spatial infinity in the spacetime that is proportional to $\mathbb{P}^\mu$,
- $\hat{\xi}^a_{\mathrm{ref}}$ is constant at infinity and on Σ, with values $\xi^a_\infty = \gamma^a_\infty$ and $\xi^a_\Sigma = \gamma^a_\infty$,
- $\hat{\xi}^0_{\mathrm{ref}}$ vanishes on Σ,
- $\hat{\xi}^i_{\mathrm{ref}} = -\beta\phi^i$ on Σ,
- $\partial_i(\hat{\xi}^0_{\mathrm{ref}})\tilde{n}^i = 16\pi\alpha$ on Σ.

We use $\tilde{n}$ to denote the unit-normal with respect to $\tilde{g}$, pointing towards infinity in M_0. Note that the condition on ξ^μ_∞ implies $\xi^\mu_\infty \mathbb{P}_\mu = m$, and the conditions on α, β and γ ensure that they correspond to the appropriate physical quantities mentioned in the statement of the Theorem.

Now for some $\xi \in W^{2,2}_{\hat{\xi}_{\mathrm{ref}}}$, define

$$\tilde{f}(G) := \mathcal{H}^\xi(G) - 16\pi\alpha\,\mathrm{Area}(\Sigma), \tag{6.48}$$

where $G = (g, A, \pi, \varepsilon) \in \mathcal{F}$. We again let $K(G) = \Phi(G) - s$, and note that for all constrained variations, $(h, b, p, f) \in \ker(DK_{\tilde{G}}) = T_{\tilde{G}}\mathcal{C}(s)$, we have (see 6.39)

$$\begin{aligned} D\mathcal{H}^{\xi}_{\tilde{G}}[h,b,p,f] =& 16\pi(\xi_\infty \cdot D\mathbb{P}'_{\tilde{G}}[h,b,p,f] + D\tilde{J}^{\xi}_{\tilde{G}}[h,b,p,f] - \xi^a_\Sigma DQ_{\Sigma\,\tilde{G}\,a}[h,b,p,f]) \\ =& 16\pi(Dm_{\tilde{G}}[h,b,p,f] - \beta DJ_{\phi\,\tilde{G}}[h,b,p,f] \\ & - \gamma_\Sigma \cdot DQ_{\Sigma\,\tilde{G}}[h,b,p,f] + \gamma_\infty \cdot DQ_{\infty\,\tilde{G}}[h,b,p,f]). \end{aligned}$$

By hypothesis (6.47), we have $D\tilde{f}_{\tilde{G}}[h,b,p,f] = 0$ for all $(h,b,p,f) \in \ker(DK_{\tilde{G}})$. It follows from Theorem 6.4, that there exists $\lambda \in \mathcal{N}$ such that

$$D\tilde{f}_{\tilde{G}} = \langle D\Phi_{\tilde{G}}, \lambda\rangle\,; \tag{6.49}$$

that is,

$$D\tilde{f}_{\tilde{G}}[h,b,p,f] = \int_{\mathcal{M}} D\Phi_{\tilde{G}}[h,b,p,f]\cdot\lambda, \tag{6.50}$$

for all $(h,b,p,f) \in T_{\tilde{G}}\mathcal{F}$. However, from Proposition 6.11, we have

$$\begin{aligned} D\tilde{f}_{\tilde{G}}[h,b,p,f] =& -\oint_\Sigma (\mathring{\nabla}^j(\xi^0)h_{ij} - \mathrm{tr}_g h\mathring{\nabla}_i(\xi^0))\sqrt{g}dS^i \\ & - \int_{\mathcal{M}_0} D\Phi^*[\xi]\cdot(h,b,p,f) - 16\pi\alpha D_{\tilde{G}}\,\mathrm{Area}(\Sigma)[h,b,p,f]. \end{aligned} \tag{6.51}$$

As $\partial_i(\xi^0)\tilde{n}^i = 16\pi\alpha$ on Σ, the first and last terms cancel exactly (see (6.37)), leaving

$$D\tilde{f}_{\tilde{G}}[h,b,p,f] = -\int_{\mathcal{M}_0} (h,b,p,f)\cdot D\Phi^*_G[\xi]; \tag{6.52}$$

that is,

$$-\int_{\mathcal{M}_0} (h,b,p,f)\cdot D\Phi^*_G[\xi] = \int_{\mathcal{M}} D\Phi_{\tilde{G}}[h,b,p,f]\cdot\lambda, \tag{6.53}$$

for all $(h,b,p,f) \in T_{(\tilde{G})}\mathcal{F}$.

Since the first integral in (6.53) is over $\mathcal{M}_0$, rather than $\mathcal{M}$, Theorem 4.10 does not directly apply. Instead we extend $D\Phi^*_{\tilde{G}}[\xi]$ by zero, noting that the hypotheses on $D\Phi^*_{\tilde{G}}[\phi]$ ensure that we can do this without losing regularity.

Define the function

$$\psi = (\psi_1, \psi_2, \psi_3, \psi_4) := \begin{cases} -D\Phi^*_{\tilde{G}}[\xi] & \text{on } \mathcal{M}_0 \\ 0 & \text{otherwise} \end{cases}. \tag{6.54}$$

We then have

$$\int_{\mathcal{M}} \psi \cdot (h, b, p, f) = \int_{\mathcal{M}} D\Phi_{\tilde{G}}[h, b, p, f] \cdot \lambda \tag{6.55}$$

for all $(h, b, p, f) \in T_{\tilde{G}}\mathcal{F}$. By Lemma 6.5, we have $\psi_1 \in L^2_{-5/2}(\mathcal{M})$ and $\psi_3, \psi_4 \in W^{1,2}_{-3/2}(\mathcal{M})$ and therefore Theorem 4.10 gives $\lambda \in W^{2,2}_{-1/2}(\mathcal{M})$ and $D\Phi^*_{\tilde{G}}[\lambda] = \psi$ in the strong sense. It then follows that $D\Phi^*_{\tilde{G}}[\tilde{\xi}] = 0$ on $\mathcal{M}_0$, where $\tilde{\xi} := \xi + \lambda$ is the generalised stationary Killing vector. □

Note that we have $D\Phi^*_{\tilde{G}}[\lambda] = 0$ on $\mathcal{M} \setminus \mathcal{M}_0$, so Theorem 4.11 implies $\lambda = 0$ on $\mathcal{M} \setminus \mathcal{M}_0$. It then follows that $\tilde{\xi} = \xi = -\beta\phi$ on Σ, and in particular we have that $\tilde{\xi} + \beta\phi^i$ vanishes on Σ. It is interesting to note that while we do not assume that Σ is a horizon in the above theorem, the conclusion that $\tilde{\xi}^\mu + \beta\phi^i$ vanishes on Σ gives us the following corollary:

Corollary 6.13. *If the hypotheses of Theorem 6.12 hold and (g, A, π, ε) is axially symmetric with axial Killing field, ϕ, then Σ is the bifurcation surface of a bifurcate Killing horizon, where $8\pi\alpha$ is the surface gravity and β is the angular velocity.*

Proof. This is an immediate consequence of the fact that if a Killing field vanishes on a spacelike 2-surface then that surface is the bifurcation surface of a bifurcate Killing horizon (see, for example, Chapter 5 of [59]). □

Remark 6.14. By virtue of the fact that $D\Phi^*[\xi] = 0$ for a Killing vector, ξ, we do indeed have $D\Phi^*_\pi[\phi], D\Phi^*_\varepsilon[\phi] \in \mathring{W}^{1,2}_{-1/2}(\mathcal{M}_0)$ when ϕ is the axial Killing vector.

6.4 Multiple Asymptotic Ends

In Section 6.2, we assumed $\mathcal{M}$ had only a single end, however a more general result holds when $\mathcal{M}$ has many asymptotic ends. In this case, the analysis is identical, however it is discussed separately for the following reasons: in this case, interpretation of the main result is not obvious, and the case of particular interest, when $\mathcal{M}$ has a single asymptotic end, is somewhat obfuscated. Note that we only include a short discussion on this case as it is not of particular interest to the thesis.

Let $\mathcal{M}$ be asymptotically flat with k ends. The constant translation near infinity, $\hat{\xi}_\infty$, may now differ between each asymptotic end. Specifically, let $\hat{\xi}_\infty$ be asymptotic to $\xi_{n\infty} \in \mathbb{R}^{3,1} \oplus \mathfrak{g}$ on the n-th end, in the same sense as the preceding Sections. Defining $\mathbb{P}_n(g, A, \pi, \varepsilon) \in \mathbb{R}^{3,1} \oplus \mathfrak{g}$ to be energy, momentum and electric charge of the n-th end, the sum of the volume integrals (5.16)-(5.18), which defined $\xi^\alpha_\infty \mathbb{P}_\alpha$ in case (a), gives $\sum_{n=1}^k \mathbb{P}_{n\,\alpha}\xi^\alpha_{n\infty}$. In this case, the appropriate energy function for Theorem 6.6 is $E^{\xi_\infty} = \sum_{n=1}^{\mathfrak{n}} \mathbb{P}_{n\,\alpha}\xi^\alpha_{n\infty}$. The space $W^{2,2}_{\xi_\infty}$ is replaced with $W^{2,2}_{\xi^N_\infty}$, of ξ asymptotic to $\xi_{n\infty}$ on each respective end. By repeating the arguments of Section 6.2 with only superficial modifications, one arrives at a slightly revised version of Theorem 6.6:

Revised Theorem 6.6. *Take $G \in \mathcal{F}$ such that $\Phi(G) = s \in L^1$. Let $\xi_\infty \in \mathbb{R}^{3,1} \oplus \mathfrak{g}$ be fixed and define the energy functional $E \in C^\infty(\mathcal{C}(c))$ by*

$$E^{\xi_\infty}(G) = \sum_{n=1}^{\mathfrak{n}} \mathbb{P}_{n\,\alpha}(G)\xi^\alpha_{n\infty}. \tag{6.56}$$

The following statements are equivalent:

(i) For all $(h, b, p, f) \in T_G\mathcal{C}(s)$,

$$DE^{\xi_\infty}_G[h, b, p, f] = 0. \tag{6.57}$$

(ii) There is $\xi \in W^{2,2}_{\xi^N_\infty}\left(\Lambda^0(\mathcal{M}) \times T\mathcal{M} \times \mathfrak{g}\right)$ *satisfying*

$$D\Phi^*_G[\xi] = 0. \tag{6.58}$$

The problem with interpreting this result, is that for many choices of the ξ_ns, neither condition, (i) or (ii), will ever be satisfied. Consider the following example: let $\xi_{n\infty} = 0$ for all $n \neq 1$, and $\xi^\mu_{1\infty}$ be a timelike vector. Then $E^{\xi_\infty} = m_1 + V_1^a Q_{1\,a}$, where m_1, V_1 and Q_1 are respectively the mass, electric potential and total electric charge viewed at infinity of a chosen end, M_1. From the revised version of Theorem 6.6, we find that solutions where $dm_1 + V_1^a dQ_{1\,a} = 0$ are those with symmetries, ξ^α, that vanish on every end except M_1. That is, ξ is asymptotic to $\xi_{n\infty} = 0$ on each end, $n \neq 1$. However, Theorem 4.11 implies that if ξ, satisfying $D\Phi^*[\xi] = 0$, is asymptotic to zero on a single end, then $\xi \equiv 0$ on $\mathcal{M}$; that is, for this choice of the $\xi_n s$, condition (ii) is never satisfied. It then follows that there is no choice of initial data satisfying $dm_1 + V_1^a dQ_{1\,a} = 0$.

6.5 Concluding Remarks

It has been demonstrated that the first law of black hole mechanics provides a condition for a solution to the Einstein-Yang-Mills equations to be stationary. In the case where we consider evolution exterior to a boundary, we do not assume that the boundary surface is a horizon; we only assume that there is a kind of symmetry of the metric and 3-potential that is tangent to the surface. We further conclude that if differential relationship pertaining to the first law is satisfied, for axially symmetric initial data, then the boundary surface is the bifurcation surface of a bifurcate Killing horizon.

In establishing these results, a suitable manifold structure for the Einstein-Yang-Mills phase space has been provided, and we have proven that the space of solutions to the constraints is a Hilbert submanifold; this is essentially equivalent to the property of linearisation stability. We also provide a Regge-Teitelboim style Hamiltonian that gives the correct equations of motion, provided that the evolution is considered away from the boundary surface.

We specifically consider the Einstein-Yang-Mills equations here as they generalise the Einstein-Maxwell equations, which is the framework for the usual first law of black hole mechanics. However, there are versions of the first law for other Einstein-matter systems and for modified gravity theories. The arguments presented here almost certainly extend in a natural way to many of these other versions of the first law.

Appendix A

Curvature Computations

In this appendix, we explicitly write the Riemann curvature in terms of g and $\mathring{g}$. This is particularly useful for obtaining the $L^2_{-5/2}$ curvature estimates used throughout (Proposition 4.2). In the following, we make use of the Christoffel symbols associated with both connections, ∇ and $\mathring{\nabla}$. We also make use of the connection difference tensor, $\tilde{\Gamma} = \Gamma - \mathring{\Gamma}$. We first write the Riemann curvature of g in terms of the Riemann curvature of $\mathring{g}$, and $\tilde{\Gamma}$:

$$\begin{aligned} R^i{}_{jkl} =& \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^i_{kn}\Gamma^n_{lj} - \Gamma^i_{ln}\Gamma^n_{kj} \\ =& \mathring{R}^i{}_{jkl} + \partial_k \tilde{\Gamma}^i_{lj} - \partial_l \tilde{\Gamma}^i_{kj} + \tilde{\Gamma}^i_{kn}\tilde{\Gamma}^n_{lj} - \tilde{\Gamma}^i_{ln}\tilde{\Gamma}^n_{kj} \\ & + \tilde{\Gamma}^i_{kn}\mathring{\Gamma}^n_{lj} + \mathring{\Gamma}^i_{kn}\tilde{\Gamma}^n_{lj} + \tilde{\Gamma}^i_{ln}\mathring{\Gamma}^n_{kj} + \mathring{\Gamma}^i_{kj}\tilde{\Gamma}^n_{kj}. \end{aligned}$$

The standard technique of fixing a point, p, and choosing coordinates such that $\mathring{\Gamma} = 0$ at p gives us the tensorial equation,

$$R^i{}_{jkl} = \mathring{R}^i{}_{jkl} + \mathring{\nabla}_k \tilde{\Gamma}^i_{lj} - \mathring{\nabla}_l \tilde{\Gamma}^i_{kj} + \tilde{\Gamma}^i_{kn}\tilde{\Gamma}^n_{lj} - \tilde{\Gamma}^i_{ln}\tilde{\Gamma}^n_{kj}, \tag{A.1}$$

which must be valid for any coordinate system and since p was arbitrary, this must hold everywhere. We now write this in terms of g:

$$\begin{aligned}
R^i{}_{jkl} =& \mathring{R}^i{}_{jkl} + \frac{1}{2}\big[\mathring{\nabla}_k(g^{im})(\mathring{\nabla}_l g_{mj} + \mathring{\nabla}_j g_{lm} - \mathring{\nabla}_m g_{lj}) \\
&+ g^{im}(\mathring{\nabla}_k\mathring{\nabla}_l g_{mj} + \mathring{\nabla}_k\mathring{\nabla}_j g_{lm} - \mathring{\nabla}_k\mathring{\nabla}_m g_{lj}) \\
&- \mathring{\nabla}_l(g^{im})(\mathring{\nabla}_k g_{mj} + \mathring{\nabla}_j g_{km} - \mathring{\nabla}_m g_{kj}) \\
&+ g^{im}(\mathring{\nabla}_l\mathring{\nabla}_k g_{mj} + \mathring{\nabla}_l\mathring{\nabla}_j g_{km} - \mathring{\nabla}_l\mathring{\nabla}_m g_{kj})\big] \\
&+ \frac{1}{4}g^{im}g^{np}(\mathring{\nabla}_k g_{mn} + \mathring{\nabla}_n g_{km} - \mathring{\nabla}_m g_{kn})(\mathring{\nabla}_l g_{pj} + \mathring{\nabla}_j g_{lp} - \mathring{\nabla}_p g_{lj}) \\
&+ \frac{1}{4}g^{im}g^{np}(\mathring{\nabla}_l g_{mn} + \mathring{\nabla}_n g_{lm} - \mathring{\nabla}_m g_{ln})(\mathring{\nabla}_k g_{pj} + \mathring{\nabla}_j g_{kp} - \mathring{\nabla}_p g_{kj}).
\end{aligned}$$

We are also interested in the expression for the scalar curvature,

$$\begin{aligned}
R =& \mathring{R} + (g^{kl} - \mathring{g}^{kl})R_{kl} + \frac{1}{2}g^{jl}\big[\mathring{\nabla}_k(g^{km})(\mathring{\nabla}_l g_{mj} + \mathring{\nabla}_j g_{lm} - \mathring{\nabla}_m g_{lj}) \\
&+ g^{km}(\mathring{\nabla}_k\mathring{\nabla}_l g_{mj} + \mathring{\nabla}_k\mathring{\nabla}_j g_{lm} - \mathring{\nabla}_k\mathring{\nabla}_m g_{lj}) \\
&- \mathring{\nabla}_l(g^{km})(\mathring{\nabla}_k g_{mj} + \mathring{\nabla}_j g_{km} - \mathring{\nabla}_m g_{kj}) \\
&+ g^{km}(\mathring{\nabla}_l\mathring{\nabla}_k g_{mj} + \mathring{\nabla}_l\mathring{\nabla}_j g_{km} - \mathring{\nabla}_l\mathring{\nabla}_m g_{kj})\big] \\
&+ \frac{1}{4}g^{jl}g^{km}g^{np}(\mathring{\nabla}_k g_{mn} + \mathring{\nabla}_n g_{km} - \mathring{\nabla}_m g_{kn})(\mathring{\nabla}_l g_{pj} + \mathring{\nabla}_j g_{lp} - \mathring{\nabla}_p g_{lj}) \\
&+ \frac{1}{4}g^{jl}g^{km}g^{np}(\mathring{\nabla}_l g_{mn} + \mathring{\nabla}_n g_{lm} - \mathring{\nabla}_m g_{ln})(\mathring{\nabla}_k g_{pj} + \mathring{\nabla}_j g_{kp} - \mathring{\nabla}_p g_{kj}).
\end{aligned}$$

Making use of the symmetry of g, we can collect terms,

$$\begin{aligned}
R =& \mathring{R} + (g^{kl} - \mathring{g}^{kl})R_{kl} + g^{km}g^{lj}\mathring{\nabla}_k\mathring{\nabla}_l g_{mj} - g^{kl}g^{mp}\mathring{\nabla}_k\mathring{\nabla}_l g_{mp} \\
&+ \frac{1}{2}g^{jl}\big[\mathring{\nabla}_k(g^{km})(\mathring{\nabla}_l g_{mj} + \mathring{\nabla}_j g_{lm} - \mathring{\nabla}_m g_{lj}) \\
&- \mathring{\nabla}_l(g^{km})(\mathring{\nabla}_k g_{mj} + \mathring{\nabla}_j g_{km} - \mathring{\nabla}_m g_{kj})\big] \\
&+ \frac{1}{4}g^{jl}g^{km}g^{np}(\mathring{\nabla}_k g_{mn} + \mathring{\nabla}_n g_{km} - \mathring{\nabla}_m g_{kn})(\mathring{\nabla}_l g_{pj} + \mathring{\nabla}_j g_{lp} - \mathring{\nabla}_p g_{lj}) \\
&+ \frac{1}{4}g^{jl}g^{km}g^{np}(\mathring{\nabla}_l g_{mn} + \mathring{\nabla}_n g_{lm} - \mathring{\nabla}_m g_{ln})(\mathring{\nabla}_k g_{pj} + \mathring{\nabla}_j g_{kp} - \mathring{\nabla}_p g_{kj}).
\end{aligned} \tag{A.2}$$

We have explicitly gathered the second order terms to draw attention to the relationship between the scalar curvature and the ADM mass. In fact, we have the expression

$$R = \mathring{\nabla}_k\big(\mathring{g}^{km}\mathring{g}^{lj}\mathring{\nabla}_l g_{mj} - \mathring{g}^{kl}\mathring{g}^{mp}\mathring{\nabla}_l g_{mp}) + o(r^{-3}), \tag{A.3}$$

where we have included $\mathring{R}$ in the $o(r^{-3})$ terms. From here, the well-known connection between the scalar curvature and ADM mass is evident. This connection provides motivation for the Regge-Teitelboim style Hamiltonian modifications in Chapter 6.

Appendix B

Bootstrapping Argument

Bootstrapping is a very useful technique used in the regularity theory of PDEs. Essentially, we have an estimate that tells us that a solution to a PDE is a little more regular than we knew a priori, then we may repeat the estimate with the new 'a priori' regularity to boost the regularity further. In this appendix, we include the remainder of the bootstrapping argument used in the proof of Theorem 4.10.

The initial estimates obtained are the following:

$$\begin{aligned}&\Gamma * F_1 \in W^{2,2}_{loc}, \qquad \Gamma * F_2 \in W^{2,3/2}_{loc}, \quad \Gamma * F_3 \in L^{3-\epsilon}_{loc}, \\ &\Gamma * \partial G_1 \in W^{1,2}_{loc}, \quad \Gamma * \partial G_2 \in W^{1,3/2}_{loc}.\end{aligned}.$$

From which, we bootstrap up to a $W^{2,2}_{loc}$ estimate. All of the estimates in this Appendix are considered on some coordinate neighbourhood, Ω.

Now, the estimate for $\Gamma * F_3$ gives us the weakest bound on u so we improve that first. Taking $\epsilon = 1/3$, we have $\xi \in L^{8/3}$, and repeating the estimates that led us here gives

$$\begin{aligned}\|\Gamma * F_3\|_{24/5} &\leq c\|\Gamma * F_3\|_{1,24/13} \leq c\|\partial\Gamma * F_3\|_{24/13} \\ &\leq c\|I_1 F_3\|_{24/13} \leq c\|F_3\|_{8/7} \leq \|\tilde{c}\|_2\|\xi\|_{8/3}.\end{aligned}$$

We have improved the regularity to $\xi \in W^{1,3/2}_{loc} \subset L^3_{loc}$, and now $\Gamma * G_2$ has apparently the weakest regularity, so we look at that next. Identical to the original estimate for $\Gamma * \partial G_1$, we have

$$\|\Gamma * \partial G_2\|_6 \leq c\|\partial(\Gamma * G_2)\|_{1,2} \leq c\|\partial^2\Gamma * G_2\|_2 \leq c\|G_2\|_2 \leq c\|b\|_6\|\xi\|_3 \leq c\|b\|_{1,2}\|\xi\|_3.$$

Now $\Gamma * F_3$ again has the weakest regularity, and we have $\xi \in W^{1,24/13}_{loc} \subset L^{24/5}_{loc}$. Repeating the above estimate we have

$$\|\Gamma * F_3\|_{24} \leq c\|\Gamma * F_3\|_{1,8/3} \leq c\|I_1 F_3\|_{8/3} \leq c\|F_3\|_{24/7} \leq \|\tilde{c}\|_2\|\xi\|_{24/5}.$$

This makes $\Gamma * \partial G_1, \Gamma * \partial G_2 \in W^{1,2}_{loc} \subset L^6_{loc}$ the terms with the weakest regularity, so we revisit their estimates. Since $\xi \in W^{1,2}_{loc}$ we have

$$\begin{aligned}\|\partial G_1\|_2 &\leq c\|\xi\|_{1,2} \\ \|\partial G_2\|_{3/2} &\leq c(\|b\partial\xi\|_{3/2} + \|\xi\partial b\|_{3/2}) \leq c(\|b\|_6\|\partial\xi\|_2 + \|\xi\|_6\|\partial b\|_2).\end{aligned}$$

The initial estimates for $\Gamma * F_1, \Gamma * F_2$ can be applied again to obtain

$$\|\Gamma * \partial G_1\|_{2,2} \leq c\|\partial G_1\|_2,$$

$$\|\Gamma * \partial G_2\|_{2,3/2} \leq c\|\partial G_2\|_{3/2}.$$

At this point we have the following regularity for each of the terms:

$$\begin{array}{lll}\Gamma * F_1 \in W^{2,2}_{loc}, & \Gamma * F_2 \in W^{2,3/2}_{loc}, & \Gamma * F_3 \in W^{1,8/3}_{loc}, \\ \Gamma * \partial G_1 \in W^{2,2}_{loc}, & \Gamma * \partial G_2 \in W^{2,3/2}_{loc}. & \end{array} .$$

Combined, we have $\xi \in W^{1,8/3}_{loc} \subset L^{24}_{loc}$. Another iteration of the $\Gamma * F_3$ estimate gives

$$\|\Gamma * F_3\|_\infty \leq c\|\Gamma * F_3\|_{1,24/5} \leq c\|I_1 F_3\|_{24/5} \leq c\|F_3\|_{24/13} \leq \|\tilde{c}\|_2\|\xi\|_{24}.$$

Note that $W^{2,3/2}$ is the critical case for the standard Sobolev inequality, so we have $\xi \in L^p_{loc}$ for all $1 \leq p < \infty$. We need one more iteration to improve this to $\xi \in L^\infty_{loc}$:

$$\|\Gamma * F_2\|_{2,2} \leq c\|F_2\|_2 \leq c\|b\|_6\|\xi\|_3 \leq c\|b\|_{1,2}\|\xi\|_3$$
$$\|\Gamma * \partial G_2\|_{2,5/3} \leq c\|\partial G_2\|_{5/3} \leq c(\|b\|_{40/9}\|\partial\xi\|_{8/3} + \|\partial b\|_2\|\xi\|_{10}).$$

Finally we have

$$\|\Gamma * \partial G_2\|_{2,2} \leq c\|\partial G_2\|_2 \leq c(\|b\|_6\|\partial\xi\|_3 + \|\partial b\|_2\|\xi\|_\infty)$$
$$\|\Gamma * F_3\|_{2,2} \leq c\|F_3\|_2 \leq c\|\tilde{c}\|_2\|\xi\|_\infty,$$

and can thus conclude $\xi \in W^{2,2}_{loc}$.

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